

## APERIODIC ORDER AND SPHERICAL DIFFRACTION

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**ABSTRACT.** We introduce model sets in arbitrary locally compact second countable (lcsc) groups, generalizing Meyer's definition of a model set in a locally compact abelian group. We then provide a new formulation of diffraction theory, which unlike the classical formulation does not involve Følner sets and thus generalizes to point sets in non-amenable lcsc groups. We focus on the case of lcsc groups admitting a Gelfand pair and on the spherical part of the diffraction. Using this approach we obtain explicit formulas for the auto-correlation and diffraction of model sets. Our diffraction formula generalizes the spherical trace formula in a similar way as the abelian diffraction formula generalizes the Poisson summation formula. We deduce that a model set has pure point spherical diffraction provided the underlying lattice is cocompact.

## 1. INTRODUCTION

Aperiodic point sets in Euclidean space are a classical object of study in geometry, combinatorics and harmonic analysis. The diffraction theory of such point sets was pioneered by Meyer already in the late 1960s [37, 38, 39]. However, it came to a wider popularity only in the 1980s, after the discovery of quasi-crystals and the subsequent attempts of physicists, crystallographers and mathematicians to provide mathematical models [29, 33, 24] explaining the icosahedral symmetry in the diffraction picture of certain aluminium-manganese alloys discovered experimentally by Shechtman et al. [50]. Diffraction theory of aperiodic point sets in locally compact abelian groups, sometimes called mathematical quasi-crystals, has remained a popular topic in abelian harmonic analysis every since. The recent monograph [1] lists several hundred references. Among these, the following were particularly influential on the current article: [13, 25, 49, 41, 3]. Further developments in the theory of mathematical quasicrystals (including diffraction theory) are covered in the works [26, 51, 31, 43, 23, 5, 4, 32, 35, 36, 47, 28, 52, 42, 2].

From the point of view of physics and crystallography, it is natural to restrict the attention to quasi-crystals in  $\mathbb{R}^n$ ,  $n \leq 3$ . From the mathematical point of view, this restriction is rather unnatural, and mathematical quasi-crystals have since long been studied in Euclidean spaces of arbitrary dimensions and in arbitrary locally compact abelian groups. However, there is no reason to stop at the class of locally compact abelian groups. In fact, we demonstrate that a large part of the theory of mathematical quasi-crystals can be carried out in the framework of arbitrary locally-compact second countable (lcsc) groups. The present article is a first attempt towards such a general theory. More specifically, the goals of this article are three-fold:

- (1) to construct plenty of examples of mathematical quasi-crystals in non-abelian (and even non-amenable) lcsc groups, and to point out some of the new phenomena which appear in this context;
- (2) to develop a theory of diffraction, which works in our general context, and specializes to the classical theory in the abelian case;
- (3) to compute in a rather explicit way the (spherical) diffraction of our examples.

**1.1. Model sets.** A special role in the Euclidean theory of quasi-crystals is played by so-called model sets, as introduced by Meyer [37, 38, 39]. These are aperiodic sets constructed by a cut-and-project scheme from lattices in products of locally compact abelian groups. In the Euclidean context one can show that a relatively dense subset  $P \subset \mathbb{R}^n$  has a uniformly discrete difference set  $P - P$  if and only if it is a subset of a model set, cf. [30, 41]. There is no analogous classification theorem for non-abelian groups, but model sets can still be defined and studied in complete analogy with the abelian case. We will use the following definition:

**Definition 1.1.** Let  $G$  and  $H$  be lcsc groups,  $\Gamma < G \times H$  a lattice and  $W_0 \subset H$  a non-empty compact subset. We denote by  $\pi_G, \pi_H$  the factor projections of the direct product  $G \times H$ . Assume that

(M1)  $\pi_G|_\Gamma$  is injective and  $\pi_H(\Gamma)$  is dense in  $H$ .

(M2)  $W_0$  is Jordan-measurable with dense interior and  $\text{Stab}_H(W_0) = \{e\}$ .

(M3)  $W_0$  is  $\Gamma$ -regular, i.e.  $\partial W_0 \cap \pi_H(\Gamma) = \emptyset$ .

Then the associated *regular model set*  $P_0 \subset G$  is given by

$$P_0 = \pi_G((G \times W_0) \cap \Gamma).$$

It is called of *compact* or *non-compact type* according to whether  $\Gamma$  is cocompact in  $G \times H$  or not.

Since we do not consider any non-regular model sets in the present article, we will drop the adjective *regular* and simply refer to sets  $P_0$  arising from the above construction as *model sets*. If  $\Gamma$  is a lattice in a product of lcsc groups  $G \times H$  satisfying (M1), then we refer to  $(G, H, \Gamma)$  as a *model set triple*, and a compact subset  $W_0 \subset H$  satisfying (M2) will be called a *window*.

**1.2. Examples of model set triples.** Not every lcsc group admits a model set; however there are plenty of examples. Here we just indicate some important classes of examples, each of which deserves a treatment of its own. We plan to return to some of these classes in future work.

- (1) *Amenable examples.* These include the classical case where  $G$  and  $H$  are abelian. A first important generalization is to nilpotent groups; if we denote by  $H(R)$  the 3-dimensional Heisenberg group over a ring  $R$ , then typical examples of nilpotent model set triples are given by  $H(\mathbb{Z}[\sqrt{2}]) < H(\mathbb{R}) \times H(\mathbb{R})$  (in analogy with the abelian example  $\mathbb{Z}[\sqrt{2}] < \mathbb{R} \times \mathbb{R}$ ) and  $H(\mathbb{Z}[1/p]) < H(\mathbb{R}) \times H(\mathbb{Q}_p)$  (in analogy with the abelian example  $\mathbb{Z}[1/p] < \mathbb{R} \times \mathbb{Q}_p$ ). Another important class of examples arises from lattices in compact extensions of nilpotent and abelian groups such as Euclidean motion groups. In this context a new phenomenon appears. As we show in Proposition 6.2 below, there exist compact-by-abelian groups  $G$  and  $H$  which admit a non-cocompact lattice  $\Gamma < G \times H$ .
- (2) *Arithmetic examples.* If  $G$  is any almost connected semisimple real Lie group with finite center and without compact factors, then there exists another such group  $H$  such that  $G \times H$  contains both a cocompact and a non-cocompact lattice satisfying (M1); for instance one can always take  $H = G$  (see Proposition 6.3). A typical (non-cocompact) example is  $\text{SL}_n(\mathbb{Z}[\sqrt{2}]) < \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{R})$ .
- (3) *S-arithmetic examples.* These are defined as their arithmetic counterparts, but for lattices in products of semisimple groups over arbitrary local fields. A typical (non-cocompact) example is  $\text{SL}_n(\mathbb{Z}[1/p]) < \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{Q}_p)$ .
- (4) *Geometric examples.* There exist examples of model sets in geometrically defined non-amenable totally-disconnected lcsc groups, such as automorphism groups of regular trees. These examples are very different in nature from the arithmetically flavored examples above. For example, while all arithmetic lattices are residually finite, Burger and Mozes [9, 10] have constructed lattices in products of automorphism groups of regular trees which are simple (and thus as far from residually finite as possible), and these lead to interesting examples of model sets.

- (5) *Hartman sets*. Yet another very different (but well-understood) class of examples arises from discrete groups  $\Gamma$  which embed densely into a compact group  $K$ . We can then view  $\Gamma$  as a lattice in  $G \times H$ , where  $G := \Gamma$  and  $H := K$  and study the corresponding model sets, which are called *Hartman sets*. We refer the interested reader to the survey [54].

**1.3. Diffraction theory in non-abelian groups.** In view of the examples above, the question arises how one can define a notion of diffraction for such general model sets. In the Euclidean case, the standard model for the diffraction of a quasi-crystal  $P_0 \subset \mathbb{R}^3$  via the theory of tempered distributions is due to Hof [25]. It works best if one assumes that for every  $f \in C_c(\mathbb{R}^3)$  the finite sums

$$\sigma_t(f) := \frac{1}{\text{Vol}(B_t(0))} \sum_{x \in P_0 \cap B_t(0)} \sum_{y \in P_0} f(y - x)$$

converge as  $t \rightarrow \infty$ , where  $B_t(0)$  denotes the Euclidean ball of radius  $t$  around the origin (as is the case for model sets). Then the diffraction  $\hat{\eta}_{P_0}$  of  $P_0$  can be defined as the Fourier transform of the auto-correlation measure  $\eta_{P_0}$ , which is given by

$$\eta_{P_0}(f) = \lim_{t \rightarrow \infty} \sigma_t(f) \quad (f \in C_c(\mathbb{R}^3)).$$

If the finite sums  $\sigma_t(f)$  do not converge, then one can consider the Fourier transforms of different accumulation points, but the significance of such arbitrary accumulation points is rather unclear.

Hof's definition of auto-correlation can be generalized to certain classes of uniformly discrete subsets of amenable lcsc groups by replacing the sequence  $B_t(0)$  by some suitable Følner sequence  $(F_t)$  in the ambient group  $G$ . Apart from the question of dependence on the Følner sequence (which can be resolved), this does yield a reasonable theory of auto-correlation, but this approach has no chance to be generalized beyond amenable groups. We thus suggest an alternative approach as follows.

Assume that  $P_0 \subset G$  is a subset of finite local complexity (i.e.  $P_0^{-1}P_0$  is closed and discrete). Let  $\mathcal{C}(G)$  denote the compact Hausdorff space of closed subsets of  $G$  with the Chabauty-Fell topology (see Appendix A) and let  $\hat{X}_{P_0}$  denote the orbit closure of  $P_0$  in  $\mathcal{C}(G)$ . We define the *hull*  $X_{P_0}$  as the complement of the empty set in  $\hat{X}_{P_0}$ . The *Siegel transform* of  $P_0$  is defined as

$$\mathcal{S} : C_c(G) \rightarrow C_b(X_{P_0}), \quad \mathcal{S}f(P) := \sum_{x \in P} f(x),$$

where  $C_b(X_{P_0})$  denotes the space of continuous bounded functions on  $X_{P_0}$ . If  $\nu$  is a  $G$ -invariant probability measure on  $X_{P_0}$ , then we define the *auto-correlation of  $P_0$  with respect to  $\nu$*  as the unique Radon measure  $\eta_{P_0, \nu}$  on  $G$  such that

$$\eta_{P_0, \nu}(f^* * f) = \|\mathcal{S}f\|_{L^2(X, \nu)}^2 \quad (f \in C_c(G)).$$

In many examples of interest, including all model sets, there is in fact a unique  $G$ -invariant measure on  $X$ ; in this case we call  $\eta_{P_0} := \eta_{P_0, \nu}$  simply the *auto-correlation* of  $P_0$ . The following theorem relates this definition to Hof's definition; for the notion of a weakly admissible Følner sequence (which is slightly more general than that of a nested van Hove sequence used e.g. in [1]), see Definition 2.8.

**Theorem 1.2** (Approximation theorem for amenable groups). *Assume that  $G$  is amenable and let  $(F_t)$  be a weakly admissible Følner sequence of compact subsets in  $G$ . If  $X_{P_0}$  is compact and uniquely ergodic, then  $\eta_{P_0}$  coincides with the classical Hof diffraction, i.e.*

$$\eta_{P_0}(f) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0} f(x^{-1}y) \quad \text{for all } f \in C_c(G). \quad (1.1)$$

Theorem 1.2 and its proof are essentially known. The assumption that  $(F_t)$  be a Følner sequence is crucial for the classical proof and prevents a direct generalization to non-amenable groups. It is thus remarkable that for all the examples of model sets in non-amenable groups listed above we can find explicit sequences  $(F_t)$  such that a version of (1.1) holds, even if the ambient group  $G$  is non-amenable and thus does not admit any Følner sequences. We will return to this point in Theorem 1.7 below.

Because of the Følner property of the  $(F_t)$ , Theorem 1.2 remains valid if one replaces the right-hand side of (1.1) by the equivalent limit

$$\lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0 \cap F_t} f(x^{-1}y).$$

However, this version does not generalize to non-amenable groups where contributions from the boundary of the  $F_t$  can no longer be neglected.

**1.4. Auto-correlation of model sets.** If  $P_0 \subset G$  is a model set associated with a quadruple  $(G, H, \Gamma, W_0)$  as above, then its hull admits a unique  $G$ -invariant probability measure  $\nu$ , and the diffraction can be computed explicitly. Note that if  $G$  is non-amenable and/or the hull is non-compact, then the mere existence of such a measure is remarkable. The key tool to establish both the existence and the uniqueness of  $\nu$  is a certain parametrization map between the hull and the parameter space  $Y := (G \times H)/\Gamma$ , which generalizes Schlottmann's torus parametrization [48, 49]. The existence of such a parametrization map yields immediately the following results; here  $m_Y$  denotes the unique  $(G \times H)$ -invariant probability measure on  $Y$ .

**Theorem 1.3** (Properties of the hull of a model set). *Let  $P_0$  be a model set and  $X = X_{P_0}$ . Then there exists a unique  $G$ -invariant probability measure  $\nu$  on  $X$ , which is also the unique stationary probability measure with respect to any admissible probability measure  $\mu$  on  $G$ . Moreover,  $(X, \nu)$  is measurably  $G$ -isomorphic to  $(Y, m_Y)$ . In particular*

$$L^2(X, \nu) \cong L^2(Y, m_Y) \quad \text{as unitary } G\text{-representations.}$$

If  $P_0$  is of compact type, then this implies in particular that  $X$  is a compact minimal  $G$ -space. However, if  $P_0$  is not of compact type, then both  $X$  and  $Y$  are non-compact, and overcoming this non-compactness is one of the major technical issues in the proof of the theorem.

From Theorem 1.3 we derive a formula for the auto-correlation of model sets in terms of the parameter space  $Y$ . Given a Riemann-integrable function  $F : G \times H \rightarrow \mathbb{C}$  with compact support we denote by

$$\mathcal{P}_\Gamma F : Y \rightarrow \mathbb{C}, \quad \mathcal{P}_\Gamma F((g, h)\Gamma) := \sum_{\gamma \in \Gamma} F((g, h)\gamma)$$

its periodization over  $\Gamma$ .

**Theorem 1.4** (Auto-correlation formula for model sets). *The auto-correlation  $\eta_{P_0}$  of a model set  $P_0$  as above is given by*

$$\eta_{P_0}(f^* * f) = \|\mathcal{P}_\Gamma(f \otimes \chi_{W_0})\|_{L^2(Y)}^2 \quad (f \in C_c(G)). \quad (1.2)$$

**1.5. Spherical auto-correlation and spherical diffraction.** By Theorem 1.4 our understanding of the auto-correlation of  $P_0$  is as good as our understanding of the  $G$ -representation  $L^2(Y)$ . In the abelian case, we can decompose  $L^2(Y)$  using the Fourier transform. For general lcsc groups  $G$ , the situation is more complicated. In order to keep this article within reasonable size, we will focus here on the case where  $G$  admits a compact subgroup  $K$  such that  $(G, K)$  is a Gelfand pair. In this case, the most accessible part of the spectrum of  $L^2(Y)$  is the *spherical spectrum*, which can be analyzed using the *spherical Fourier transform* of the pair  $(G, K)$ .

The basic theory of the spherical Fourier transform of a Gelfand pair  $(G, K)$  is summarized in Appendix B. The two basic objects underlying the spherical Fourier transform are the Hecke

algebra  $\mathcal{H}(G, K)$  of the pair  $(G, K)$ , i.e. the space of bi- $K$ -invariant compactly supported continuous functions on  $G$  with the convolution product, and the space  $\mathcal{S}^+(G, K)$  of positive definite  $K$ -spherical functions on  $G$  with its natural locally-compact topology (see Subsection B.4). The spherical Fourier transform associates to every  $f \in \mathcal{H}(G, K)$  a function  $\hat{f} \in C_0(\mathcal{S}^+(G, K))$  (see Definition B.2) and to every every positive definite linear functional  $\mu$  on  $\mathcal{H}(G, K)$  a positive Radon measure  $\hat{\mu}$  on  $\mathcal{S}^+(G, K)$  (see Subsection B.6).

We define the *spherical diffraction*  $\hat{\eta}_{P_0}$  of a model set  $P_0 \subset G$  as the spherical Fourier transform of its auto-correlation measure  $\eta_{P_0}$  (restricted to the Hecke algebra). It turns out that a model set has pure point spherical diffraction provided it is of compact type. In this case we can give an explicit formula for  $\hat{\eta}_{P_0}$  as follows. Assume that  $\Gamma$  is cocompact and let

$$\Omega_Y := \{\omega \in \mathcal{S}^+(G, K) \mid \exists g \in L^2(Y)^K \setminus \{0\} \forall f \in \mathcal{H}(G, K): f * g = \hat{f}(\omega) \cdot g\}$$

denote the *spherical spectrum* of  $Y$ . Given  $\omega \in \Omega_Y$ , define

$$L^2(Y)_\omega^K = \{g \in L^2(Y)^K \mid \forall f \in \mathcal{H}(G, K): f * g = \hat{f}(\omega) \cdot g\}.$$

as the associated weight space and write  $\pi_\omega : L^2(Y)^K \rightarrow L^2(Y)_\omega^K$  for the orthogonal projection. The main new ingredient in our general diffraction formula is a certain integral transform which can be characterized as follows.

**Lemma 1.5.** *There exists a unique map  $\mathbb{S} : C_c(H) \rightarrow \text{Map}(\Omega_Y, L^2(Y))$ ,  $r \mapsto \hat{r}$  such that if  $\omega \in \Omega_Y$  and  $f \in \mathcal{H}(G, K)$  with  $\hat{f}(\omega) = 1$ , then*

$$\hat{r}(\omega) = \pi_\omega(\mathcal{P}_\Gamma(f \otimes r)).$$

We refer to  $\mathbb{S} : C_c(H) \rightarrow \text{Map}(\Omega_Y, L^2(Y))$  as the *shadow transform*. It can be extended to Riemann-integrable functions by the usual approximation argument. We discuss basic properties of the shadow transform in Subsection 4.2. In particular, we show that  $r \in C_c(H)$  lies in the kernel of the shadow transform if and only if the periodization of  $f \otimes r$  over  $\Gamma$  vanishes for every  $f \in \mathcal{H}(G, K)$ .

Two instances of the shadow transform have been studied before in the literature: When  $K$  is trivial (and hence  $G$  and  $H$  are abelian), then the shadow transform is closely related to the Fourier transform of  $H$ . On the other hand, if  $K$  is open in  $G$ , then the shadow transform is closely related to the associated Hecke correspondence (see Subsection 4.4). In these cases, the kernel of the shadow transform is always trivial, but we do not know whether this holds in general.

**Theorem 1.6** (Spherical diffraction of model sets of compact type). *If  $P_0$  is a model set of compact type, then*

$$\hat{\eta}_{P_0} = \sum_{\omega \in \Omega_Y} \|\widehat{\chi_{W_0}}(\omega)\|_{L^2(Y)}^2 \cdot \delta_\omega.$$

In the abelian case, the diffraction formula reduces to a classical formula of Meyer (see [40, Thm. 7], also [1, Thm. 9.4]), which specializes further to the Poisson summation formula in the case of lattices, and hence is sometimes referred to as an “aperiodic Poisson summation formula”. In a similar vein, the above diffraction formula is an “aperiodic spherical trace formula”. As in the case of the classical trace formula, one can extend this formula to deal with the complete spectrum (rather than just the spherical one) and/or with model sets of non-cocompact type. In the latter case, there will be a contribution from the continuous spectrum, hence the sum in Theorem 1.6 has to be replaced by an integral, and the diffraction is no longer pure point. We plan to cover this case more explicitly in future work.

**1.6. Approximation of the spherical auto-correlation.** We now return to the question of approximating the auto-correlation by finite sums in the setting of Gelfand pairs. Of particular interest in this context is the *spherical auto-correlation*, i.e. the restriction of the auto-correlation to  $\mathcal{H}(G, K)$ , since it determines the spherical diffraction. For the spherical auto-correlation we have the following version of Theorem 1.2.

**Theorem 1.7** (Approximation theorem for the spherical auto-correlation). *Let  $(G, K)$  be a Gelfand pair and  $P_0 \subset G$  be a model set with underlying lattice  $\Gamma$ . If  $(F_t)$  is any good approximation sequence for  $\Gamma$ , then the spherical auto-correlation of  $P_0$  is given by*

$$\eta_{\text{sph}}(f) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0} f(x^{-1}y) \quad (f \in \mathcal{H}(G, K)).$$

Here a sequence  $(F_t)$  of bi- $K$ -invariant subsets of  $G$  is called a *good approximation sequence* for  $\Gamma$  if it is weakly admissible (see Definition 2.8) and satisfies a certain spectral condition with respect to  $\Gamma$ , which is explicited in Definition 5.8. If  $G$  is amenable, then this spectral condition is satisfied by any Følner sequence  $(F_t)$ , but good approximation sequences (unlike Følner sequences) exist in many non-amenable groups. For instance, if  $G$  is a semisimple  $S$ -adic group, then the spectral condition holds automatically, and thus any weakly admissible sequence of bi- $K$ -invariant subsets of  $G$  is a good approximation sequence for any  $\Gamma$  (see Proposition 6.4).

A particularly interesting example is given by model sets in the automorphism group of a regular tree  $T$  (of even valency  $\geq 30$ ) which arise from Burger–Mozes lattices. In this case, a natural sequence  $(F_n)_{n \in \mathbb{N}}$  of balls in  $G := \text{Aut}(T)$  is given by pre-images of balls in the tree around some baspoint  $o$  under the orbit map. If we denote by  $K$  the stabilizer of  $o$ , then  $(G, K)$  is a Gelfand pair and one can find  $f \in \mathcal{H}(G, K)$  such that the sequence

$$\frac{1}{m_G(F_n)} \sum_{x \in P_0 \cap F_n} \sum_{y \in P_0} f(x^{-1}y)$$

does *not* converge. On the other hand, it follows from work of Lubotzky and Mozes [34] that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{m_G(F_{2n})} \sum_{x \in P_0 \cap F_{2n}} \sum_{y \in P_0} f(x^{-1}y)$$

over the *even* balls exists for all  $f \in \mathcal{H}(G, K)$  (and equals the auto-correlation), see Proposition 6.6. This can be seen as a variant of the Nevo-Stein phenomenon [44], which has no counterpart in the classical theory of abelian model sets.

**1.7. Organization of the article.** This article is organized as follows: In Section 2 we discuss our general framework for diffraction theory in lcsc groups via Siegel transforms and compare it to the classical version for abelian groups. In Section 3 we introduce model sets and study their hulls through a generalization of Schlottmann’s torus parametrization. In particular, we establish Theorem 1.3. Section 4 is the core of the present article in which we establish our main results, Theorem 1.4 and Theorem 1.6. Section 5 is entirely concerned with the proof of Theorem 1.7. The final section, Section 6, is devoted to various classes of examples.

For the convenience of the reader we have collected some background material in two appendices. Appendix A describes basic properties of the Chabauty–Fell topology on various spaces of closed subsets of a lcsc group, some of which were previously only available in the abelian case. Appendix B is a brief survey on those properties of Gelfand pairs which are relevant for this article.

**1.8. Notational conventions.** Throughout this article we use the following conventions. All function spaces are complex-valued and all inner products are anti-linear in the first variable. Given a locally compact space  $X$  we denote by  $C_c(X)$ ,  $C_0(X)$  and  $C_b(X)$  the function space of compactly supported continuous functions, continuous functions vanishing at infinity and continuous bounded functions respectively.

Given a group  $G$  and a function  $f : G \rightarrow \mathbb{C}$  we denote by  $\bar{f}$ ,  $\check{f}$  and  $f^*$  respectively the functions on  $G$  given by  $\bar{f}(g) := \overline{f(g)}$ ,  $\check{f}(g) := f(g^{-1})$  and  $f^*(g) := \overline{f(g^{-1})}$ . Given an action of  $G$  on a set  $Z$  we define a  $G$ -action on complex-valued functions on  $Z$  by  $g.f(z) := f(g^{-1}.z)$ . Moreover we denote by  $Z^G \subset Z$  the subset of  $G$ -invariants.

If  $(X, \nu)$  is a measure space and  $f, g \in L^2(X, \nu)$ , then we denote by  $\langle f, g \rangle_X := \langle f, g \rangle_{(X, \nu)} := \int_X \bar{f} \cdot g d\nu$  the  $L^2$ -inner product. We also write  $L_0^2(X, \nu) \subset L^2(X, \nu)$  for the orthogonal complement of the constant functions. Given a subset  $A \subset X$  we denote by  $\chi_A$  its characteristic function.

If  $G$  is a locally compact, second countable group, then we denote by  $m_G$  some fixed choice of left-Haar measure on  $G$  (normalized to total mass 1 in the compact case). We then write  $\mathcal{C}(G)$  and  $\mathcal{K}(G)$  for the collections of closed and compact subsets of  $G$  respectively and denote by  $\mathcal{U}(G)$  the identity neighbourhood filter of  $G$ . We equip  $\mathcal{C}(G)$  with the Chabauty-Fell topology (see Appendix A), thereby turning it into a compact Hausdorff space.

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## 2. A GENERAL FRAMEWORK FOR DIFFRACTION

**2.1. Orbit closures of finite local complexity sets.** Let  $G$  be a locally compact, second countable (lcsc) group. A subset  $P_0 \subset G$  is said to have *finite local complexity* (abbreviated FLC) if  $P_0^{-1}P_0 \subset G$  is closed and discrete. Recall that  $\mathcal{C}(G)$  denotes the compact Hausdorff space of closed subsets of  $G$  with the Chabauty-Fell topology.

**Convention 2.1.** Throughout this article,  $G$  denotes a locally compact, second countable (lcsc) group, and  $P_0 \subset G$  denotes a subset of finite local complexity. We write

$$\widehat{X} := \widehat{X}_{P_0} := \overline{G.P_0} \subset \mathcal{C}(G)$$

for the orbit closure of  $P_0$  in  $\mathcal{C}(G)$ . The subspace

$$X := X_{P_0} := \widehat{X} \setminus \{\emptyset\} \subset \widehat{X}$$

is called the *hull* of  $P_0$ .

In the abelian context it is customary to assume that  $P_0$  is relatively dense, which implies that  $\widehat{X} = X$  (see Corollary A.18). In the non-amenable setting, there are many natural examples of FLC sets (in particular, model sets constructed from non-uniform lattices) which fail to satisfy this assumption. In order to be able to treat these examples we have to deal with the case where  $X \subsetneq \widehat{X}$  is non-compact, and in particular to carefully distinguish between  $\widehat{X}$  and  $X$ .

**Proposition 2.2** (Orbit closures of FLC sets). *For  $\widehat{X} = \widehat{X}_{P_0}$  the following hold.*

- (i)  $\widehat{X}$  is a compact Hausdorff space, and the  $G$ -action on  $\widehat{X}$  is jointly continuous.

(ii) For every  $P \in \widehat{X}$  the neighbourhood filter of  $P$  in  $\widehat{X}$  is generated by the sets

$$U_{K,V}(P) = \{Q \in \widehat{X} \mid \exists t \in V : tQ \cap K = P \cap K\},$$

where  $K$  ranges over  $\mathcal{K}(G)$  and  $V$  ranges over  $\mathfrak{U}(G)$ .

(iii) For every  $P \in \widehat{X}$  we have  $P^{-1}P \subset P_0^{-1}P_0$ . In particular, every  $P \in \widehat{X}$  has finite local complexity.

(iv) If  $\Gamma_G$  is the subgroup of  $G$  generated by  $P_0$  and  $P \in \widehat{X}$ , then  $p^{-1}P \subset \Gamma_G$  for every  $p \in P$ .

(v) For every  $K \in \mathcal{K}(G)$  there exists  $C_K > 0$  such that for all  $P \in \widehat{X}$  and  $g \in G$ ,

$$|P \cap gK| \leq C_K.$$

*Proof.* (i) Proposition A.1. (ii) Corollary A.11. (iii) Proposition A.12. (iv) is immediate from (iii). (v) Lemma A.14.  $\square$

**2.2. The Siegel transform.** Let  $P_0, G$  and  $X$  as in Convention 2.1.

**Proposition 2.3** (Continuity of the Siegel transform). *There is a well-defined continuous  $G$ -equivariant map*

$$\mathcal{S} : C_c(G) \rightarrow C_b(X), \quad \mathcal{S}f(P) := \sum_{g \in P} f(g).$$

*Proof.* Let  $f \in C_c(G)$  and  $K := \text{supp}(f)$ . By Proposition 2.2.(iii), every  $P \in X$  has FLC, hence is locally finite (i.e. closed and discrete). It follows that the sum

$$\sum_{g \in P} f(g)$$

is finite, and consequently  $\mathcal{S}f : X \rightarrow \mathbb{C}$  is well-defined. Since

$$g.(\mathcal{S}f)(P) = \mathcal{S}f(g^{-1}.P) = \sum_{x \in g^{-1}.P} f(x) = \sum_{x \in P} f(g^{-1}.x) = \sum_{x \in P} (g.f)(x) = \mathcal{S}(g.f)(P),$$

the map  $f \mapsto \mathcal{S}f$  is  $G$ -equivariant. To see that  $\mathcal{S}f$  is bounded, recall from Proposition 2.2 there exists a constant  $C > 0$  such that for all  $P \in X$  and  $g \in G$  we have  $|gK \cap P| \leq C$ . Then for all  $P \in X$ ,

$$|\mathcal{S}f(P)| \leq \sum_{x \in P} |f(x)| \leq |K \cap P| \cdot \|f\|_\infty \leq C \cdot \|f\|_\infty,$$

hence  $\mathcal{S}f$  is bounded.

To see that  $\mathcal{S}f$  is continuous, let  $P_n$  be a sequence in  $X$  which converges to  $P \in X$ . Let  $\epsilon > 0$  and let  $V \in \mathfrak{U}(G)$  be pre-compact and symmetric such that  $|f(tx) - f(x)| < \epsilon/C$  for all  $t \in V$  and  $x \in G$ . By the description of the topology on  $X$  in Proposition 2.2, there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  there exists  $t_n \in V$  such that

$$t_n P_n \cap VK = P \cap VK.$$

Then

$$\begin{aligned} |\mathcal{S}f(P_n) - \mathcal{S}f(P)| &= \left| \sum_{x \in P_n \cap K} f(x) - \sum_{x \in P \cap K} f(x) \right| = \left| \sum_{x \in P_n \cap K} f(x) - \sum_{x \in t_n P_n \cap K} f(x) \right| \\ &= \left| \sum_{x \in P_n} f(x) - \sum_{x \in P_n} f(t_n x) \right| \leq \sum_{x \in P_n} |f(x) - f(t_n x)| \\ &< |P_n \cap K| \cdot \epsilon/C \leq \epsilon, \end{aligned}$$

hence  $\mathcal{S}f(P_n) \rightarrow \mathcal{S}f(P)$ , showing that  $\mathcal{S}f \in C_b(X)$ .

Finally, assume that  $f_n \rightarrow f$  in  $C_c(G)$  and let  $\epsilon > 0$ . We may assume that for some compact set  $K$  we have  $\text{supp}(f_n) \subset K$  for all  $n \in \mathbb{N}$  and  $\text{supp}(f) \subset K$ . Let again  $C > 0$  be such that for all  $P \in X$



and  $g \in G$  we have  $|gK \cap P| \leq C$ . For large  $n$  we then have  $\|f_n - f\|_\infty < \epsilon/C$  and thus for every  $P \in X$

$$|\mathcal{S}(f_n)(P) - \mathcal{S}(f)(P)| \leq \sum_{x \in P} |f(x) - f_n(x)| \leq |P \cap K| \cdot \epsilon/C \leq \epsilon,$$

hence  $\|\mathcal{S}(f_n) - \mathcal{S}(f)\|_\infty < \epsilon$ . This finishes the proof.  $\square$

**Definition 2.4.** The map  $\mathcal{S} : C_c(G) \rightarrow C_b(X_{P_0})$  is called the *Siegel transform* of  $P_0$ .

**2.3. The auto-correlation measure of an FLC set.** If the hull of an FLC set  $P_0$  admits a  $G$ -invariant probability measure, then we can define an autocorrelation measure for  $P_0$  as follows. Note however that the existence of such a measure is a non-trivial condition if  $G$  is non-amenable and/or  $X$  is non-compact.

**Proposition 2.5** (Construction of the autocorrelation measure). *Assume that  $P_0 \subset G$  is of finite local complexity and that the hull  $X = X_{P_0}$  admits a  $G$ -invariant probability measure  $\nu$ . Then there exists a unique positive definite Radon measure  $\eta_\nu \in \mathcal{R}(G)$  such that*

$$\eta_\nu(f^* * f) = \|\mathcal{S}f\|_{L^2(X, \nu)}^2 \quad (f \in C_c(G)). \quad (2.1)$$

*Proof.* We can define a Radon measure  $\tilde{\eta}_\nu$  on  $G \times G$  by

$$\tilde{\eta}_\nu(\overline{f_1} \otimes f_2) := \langle \mathcal{S}f_1, \mathcal{S}f_2 \rangle_{L^2(X, \nu)} = \int_X \overline{\mathcal{S}f_1} \cdot \mathcal{S}f_2 \, d\nu \quad (f_1, f_2 \in C_c(G)).$$

Note that if  $f_1, f_2 \in C_c(G)$  then  $\mathcal{S}f_1, \mathcal{S}f_2 \in C_b(X) \subset L^2(X, \nu)$ , so  $\tilde{\eta}_\nu$  is indeed well-defined. Also note that  $\tilde{\eta}_\nu$  is invariant under the diagonal  $G$ -action on  $G \times G$ , since the measure  $\nu$  is assumed  $G$ -invariant and the Siegel transform is  $G$ -equivariant. We deduce that  $\tilde{\eta}_\nu$  descends to a measure on  $\Delta(G) \backslash (G \times G)$ , where  $\Delta(G)$  denotes the diagonal subgroup. Under the homeomorphism  $\Delta(G) \backslash (G \times G) \rightarrow G$  sending  $[g, h]$  to  $g^{-1}h$  this measure  $\tilde{\eta}_\nu$  then corresponds to a measure  $\eta_\nu \in \mathcal{R}(G)$ , which is given by (2.1). Indeed, for all  $f_1, f_2 \in C_c(G)$ , we have  $\tilde{\eta}_\nu(\overline{f_1} \otimes f_2) = \eta_\nu(f_{12})$ , where

$$\begin{aligned} f_{12}(g^{-1}h) &= \int_G (\overline{f_1} \otimes f_2)(rg, rh) \, dm_G(r) = \int_G \overline{f_1}(r) f_2(rg^{-1}h) \, dm_G(r) \\ &= \int_G f_1^*(r) f_2(r^{-1}g^{-1}h) \, dm_G(r) = (f_1^* * f_2)(g^{-1}h), \quad \text{for all } g, h \in G. \end{aligned}$$

Since  $\eta_\nu(f^* * f) = \|\mathcal{S}f\|_{L^2(X, \nu)}^2 \geq 0$  for all  $f$ , the measure  $\eta_\nu$  is positive definite.  $\square$

**Definition 2.6.** The measure  $\eta_\nu$  defined by Proposition 2.5 is called the *auto-correlation measure* of  $P_0$  with respect to  $\nu$ .

In most of our cases of interest, the hull of  $P_0$  will actually be *uniquely ergodic*, i.e. admit a unique invariant probability measure  $\nu$ . In this case we refer to  $\eta = \eta_\nu$  similar as *the* auto-correlation measure of  $P_0$ .

**Remark 2.7.** In the situation of Proposition 2.5 we can define a whole family of Radon measures  $\tilde{\eta}_\nu^{(n)}$  on  $G^n$  by

$$\tilde{\eta}_\nu^{(n)}(f_1 \otimes \cdots \otimes f_n) := \int_X \mathcal{S}f_1 \cdots \mathcal{S}f_n \, d\nu \quad (f_1, \dots, f_n \in C_c(G)).$$

By the same argument as in the proof of Proposition 2.5, all these Radon measures are invariant under the diagonal  $G$ -action, and hence descend to *higher correlation measures*  $\eta^{(n)} \in \mathcal{R}(G^{n-1})$ .

For  $n = 1$  we conclude that  $\tilde{\eta}_\nu^{(1)}$  is a left-Haar measure. Comparing this measure to our fixed left-Haar measure  $m_G$  we find that there exists a constant  $h = h_{P_0, \nu, m_G} > 0$  such that

$$\int_X \mathcal{S}f \, d\nu = h_{P_0, \nu, m_G} \cdot \int_G f \, dm_G \quad (f \in C_c(G)).$$

We call  $h_{P_0, \nu, m_G}$  the *Siegel constant* of  $P_0$  with respect to  $\eta$  and  $m_G$ .

In the case where  $\nu$  is an ergodic measure on  $X$ , the Siegel constant  $h_{P_0, \nu, m_G}$  can actually be recovered from the auto-correlation measure  $\eta_\nu$ . More precisely, we claim that if  $\rho \in C_c(G)$  with  $\rho = \rho^*$ ,  $\rho \geq 0$  and  $\int \rho dm_G = 1$ , and if moreover  $\text{supp}(\rho)$  generates  $G$ , then

$$h_{P_0, \nu, m_G} = \sqrt{\lim_{n \rightarrow \infty} \eta_\nu(\rho^{*n})}. \quad (2.2)$$

*Proof of (2.2).* Let  $h := h_{P_0, \nu, m_G}$ . Observe that  $f_\rho := \mathcal{S}\rho - h$  is square-integrable and satisfies  $\int f_\rho dm_G = 0$  by definition of  $h$ . Thus  $f_\rho \in L^2_o(X, \nu)$ , and since  $(X, \nu)$  is assumed ergodic it follows from [27, Thm. 3.11] that

$$\langle f_\rho, \pi_X(\rho)^n f_\rho \rangle_X \rightarrow 0.$$

On the other hand we have

$$\begin{aligned} |\langle f_\rho, \pi_X(\rho)^n f_\rho \rangle_X| &= |\langle \mathcal{S}\rho - h, \rho^{*n} * \mathcal{S}\rho - h \rangle_X| \\ &= |\langle \mathcal{S}\rho, \rho^{*n} * \mathcal{S}\rho \rangle_X - \langle \mathcal{S}\rho, h \rangle_X - \langle h, \rho^{*n} * \mathcal{S}\rho \rangle_X + h^2| \\ &= |\langle \mathcal{S}\rho, \rho^{*n} * \mathcal{S}\rho \rangle_X - h^2 - h^2 + h^2| = |\langle \mathcal{S}\rho, \mathcal{S}\rho^{*(n+1)} \rangle_X - h^2| \\ &= |\eta(\rho^{*(n+2)}) - h^2|, \end{aligned}$$

and thus  $\eta(\rho^{*(n+2)}) \rightarrow h^2$ , finishing the proof.  $\square$

**2.4. An approximation theorem for the auto-correlation.** Throughout this subsection we fix a  $G$ -invariant probability measure  $\nu$  on the hull  $X$  of  $P_0$  and denote by  $\eta_\nu$  the associated auto-correlation measure. The goal of this subsection is to find conditions on a sequence  $(F_t)$  of subsets of  $G$  and a class of functions  $\mathcal{A} \subset C_c(G)$  such that the finite sums

$$\sigma_t(f) := \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0} f(x^{-1}y) \quad (2.3)$$

approximate  $\eta_\nu(f)$  for all  $f \in \mathcal{A}$ .

To formulate the conditions on the sequence  $(F_t)$ , we fix a left-invariant metric on  $G$  and denote by  $B_\delta$  the open ball of radius  $\delta$  around the identity. Given a subset  $L \subset G$  we then denote

$$L_\delta = LB_\delta \quad \text{and} \quad L_{-\delta} = \bigcap_{t \in B_\delta} Lt.$$

The following definition is a weakening of the notion of an *admissible sequence* from [18].

**Definition 2.8.** We say that a sequence  $(F_t)$  of compact subsets of  $G$  is *weakly admissible* if each  $(F_t)$  has positive Haar measure and there are continuous functions  $\alpha, \beta : [0, 1) \rightarrow \mathbb{R}_+$  with  $\alpha(0) = 0$  and  $\beta(0) = 0$  such that

$$(F_t)_\delta \subset F_{t+\alpha(\delta)} \quad \text{and} \quad \sup_s \frac{m_G(F_{s+\delta})}{m_G(F_s)} = 1 + \beta(\delta),$$

for all  $t, \delta > 0$ . We shall refer to the pair  $(\alpha, \beta)$  as the *parameters* of  $(F_t)$ .

Concerning the class of functions  $\mathcal{A}$  we are going to assume the following condition.

**Definition 2.9.** Let  $(F_t)$  be a sequence of compact subsets of  $G$  of positive Haar measures. We say that a linear sub-space  $\mathcal{A} \subset C_c(G)$  is *generic* with respect to  $\nu$  and the sequence  $(F_t)$  if

$$\eta_\nu(f_1^* * f_2) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \int_{F_t} \overline{\mathcal{S}f_1(s^{-1} \cdot P_0)} \mathcal{S}f_2(s^{-1} \cdot P_0) dm_G(s) \quad \text{for all } f_1, f_2 \in \mathcal{A}.$$

Now we can state the desired approximation theorem.

**Theorem 2.10** (Abstract approximation theorem). *Suppose that  $(F_t)$  is a weakly admissible sequence of compact subsets of  $G$  and that  $\mathcal{A} \subset C_c(G)$  is generic with respect to  $\nu$  and  $(F_t)$ . Then for every  $f \in \mathcal{A}$  the finite sums (2.3) converge to the auto-correlation, i.e.*

$$\eta_\nu(f) = \lim_{t \rightarrow \infty} \sigma_t(f).$$

As a special case we obtain:

**Corollary 2.11** (Approximation theorem for amenable groups). *Assume that  $G$  is amenable and let  $(F_t)$  be a weakly admissible Følner sequence of compact subsets in  $G$ . If  $X$  is compact and uniquely ergodic, then*

$$\eta_\nu(f) = \lim_{t \rightarrow \infty} \sigma_t(f) \quad \text{for all } f \in C_c(G).$$

The proofs of Theorem 2.10 and Corollary 2.11 will be given in the following subsection. We will see in Section 5 below that Theorem 2.10 applies way beyond amenable groups. We emphasize that no compactness assumption on  $X$  is made in Theorem 2.10.

**2.5. The proof of the approximation theorem.** The proofs of Theorem 2.10 and its corollary are based on the following observations:

**Lemma 2.12.** *Suppose that  $\psi$  is a non-negative left uniformly continuous function on  $G$ . Then, for every  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that*

$$\sum_{x \in P_0 \cap L_{-\delta}} \psi(x) - \varepsilon |P_0 \cap L_{-\delta}| \leq \int_L (\mathcal{S}\rho)(s^{-1}P_0)\psi(s) dm_G(s) \leq \sum_{x \in P_0 \cap L_\delta} \psi(x) + \varepsilon |P_0 \cap L_\delta|$$

for every compact set  $L \subset G$  and non-negative  $\rho \in C_c(G)$  with  $\text{supp } \rho \subset B_\delta$  and  $\int_G \rho dm_G = 1$ .

**Lemma 2.13.** *For every  $\delta > 0$ , there exists a constant  $M_\delta$  such that*

$$M_\delta \geq \frac{|P_0 \cap L_{-\delta}|}{m_G(L)} \quad \text{for every compact subset } L \subset G.$$

Applying this to weakly admissible sequences we obtain:

**Corollary 2.14.** *For every weakly admissible sequence  $(F_t)$ , we have*

$$\sup_t \frac{|P_0 \cap F_t|}{m_G(F_t)} < \infty.$$

Let us first explain how Lemma 2.12 and Corollary 2.14 imply Theorem 2.10.

*Proof of Theorem 2.10.* Fix  $f \in \mathcal{A}$ , which we may assume is non-negative, and note that  $\psi(s) = \mathcal{S}f(s^{-1} \cdot P_0)$  is non-negative and left-uniformly continuous on  $G$ . Let  $(F_t)$  be a weakly admissible sequence of subsets in  $G$  with associated parameters  $(\alpha, \beta)$ . We wish to prove that

$$\eta_\nu(f) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \psi(x).$$

Fix  $\varepsilon > 0$  and choose  $\delta > 0$  as in Lemma 2.12 so that for every non-negative continuous function  $\rho$  supported on  $B_\delta$  with  $\int_G \rho dm_G = 1$  and every  $t \in \mathbb{R}$  we have

$$\sum_{x \in P_0 \cap (F_t)_{-\delta}} \psi(x) - \varepsilon |P_0 \cap (F_t)_\delta| \leq \int_{F_t} (\mathcal{S}\rho)(s^{-1}P_0)\psi(s) dm_G(s) \leq \sum_{x \in P_0 \cap (F_t)_\delta} \psi(x) + \varepsilon |P_0 \cap (F_t)_\delta|.$$

If we define

$$\Xi_\rho(t) = \frac{1}{m_G(F_t)} \int_{F_t} (\mathcal{S}\rho)(s^{-1}P_0)\psi(s) dm_G(s),$$

then, since  $\mathcal{A}$  is assumed to be generic with respect to  $\nu$  and  $(F_t)$ , we have

$$\eta_\nu(\rho^* * f) = \lim_{t \rightarrow \infty} \Xi_\rho(t).$$

Since  $(F_t)$  is weakly admissible, we have for all  $t > 0$ ,

$$F_{t-\alpha(\delta)} \subset (F_t)_{-\delta} \quad \text{and} \quad (F_t)_\delta \subset F_{t+\alpha(\delta)},$$

and

$$\frac{m_G(F_{t-\alpha(\delta)})}{m_G(F_t)} \geq \frac{1}{1+\beta(\delta)} \quad \text{and} \quad \frac{m_G(F_{t+\alpha(\delta)})}{m_G(F_t)} \leq 1+\beta(\delta),$$

Moreover, by Corollary 2.14,

$$M := \sup_t \frac{|P_0 \cap F_t|}{m_G(F_t)} < \infty.$$

Hence, if we define

$$\Psi(t) = \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \psi(x), \quad \Psi_- = \varliminf_t \Psi(t) \quad \text{and} \quad \Psi_+ = \overline{\varliminf_t \Psi(t)},$$

then it follows that for all  $t > 0$

$$\frac{1}{1+\beta(\delta)} \Psi(t - \alpha(\delta)) - \varepsilon(1+\beta(\delta))M \leq \Xi_\rho(t) \leq (1+\beta(\delta))(\Psi(t) + \varepsilon M),$$

and thus in particular

$$\frac{1}{1+\beta(\delta)} \Psi_+ - \varepsilon(1+\beta(\delta))M \leq \eta_\nu(\rho^* * f) \leq (1+\beta(\delta))(\Psi_- + \varepsilon M).$$

Note that these estimates are uniform in  $\varepsilon$ . We may now choose a decreasing sequence  $(\varepsilon_n)$  which converges to zero, and pick  $\delta_n$  and  $\rho_n$  correspondingly. Since  $f$  has compact support and  $\eta_\nu$  is finite on compact subsets of  $G$ , we have

$$\lim_{n \rightarrow \infty} \eta_\nu(\rho_n^* * f) = \eta_\nu(f),$$

and thus, since  $\beta$  is continuous and  $\beta(0) = 0$ , we have

$$\Psi_+ \leq \eta_\nu(f) \leq \Psi_-.$$

This shows that  $\Psi_+ = \Psi_- = \eta_\nu(f)$ , and thus finishes the proof.  $\square$

We now turn to the proofs of the lemmas.

*Proof of Lemma 2.12.* Let  $\psi$  be a left uniformly continuous function on  $G$ . Fix  $\varepsilon > 0$  and choose  $\delta > 0$  such that

$$|\psi(s) - \psi(x)| < \varepsilon, \quad \text{for all } s, x \in G \text{ such that } s^{-1}x \in B_\delta. \quad (2.4)$$

Let  $\rho$  be a non-negative continuous function on  $G$  supported on  $B_\delta$  with  $\int_G \rho dm_G = 1$  and  $L \subset G$  be a compact set. Note firstly that if  $s^{-1}x \in B_\delta$ ,  $s \in L$ ,  $x \in P_0$  and then  $x \in P_0 \cap L_\delta$ . This implies that

$$\begin{aligned} \int_L \mathcal{S} \rho(s^{-1} \cdot P_0) \psi(s) dm_G(s) &= \sum_{x \in P_0} \int_L \rho(s^{-1}x) \psi(s) dm_G(s) \\ &= \sum_{x \in P_0 \cap L_\delta} \int_L \rho(s^{-1}x) (\psi(s) - \psi(x)) dm_G(s) \\ &\quad + \sum_{x \in P_0 \cap L_\delta} \left( \int_L \rho(s^{-1}x) dm_G(s) \right) \psi(x). \end{aligned}$$

By the relation between  $\psi$  and  $B_\delta$  described in (2.4), and by the bound  $\int_L \rho(s^{-1}x) dm_G(s) \leq 1$  for all  $x \in G$ , we see that

$$\int_L \mathcal{S} \rho(s^{-1} \cdot P_0) \psi(s) dm_G(s) \leq \varepsilon |P_0 \cap L_\delta| + \sum_{x \in P_0 \cap L_\delta} \psi(x),$$

which finishes the proof of the upper bound. Concerning the lower bound, we observe that if  $x \in L_{-\delta}$ , then  $L^{-1}x \supset B_\delta$  and thus

$$\int_L \rho(s^{-1}x) dm_G(s) = 1 \quad \text{for all } x \in L_{-\delta}.$$

Combining this with (2.4) we conclude that

$$\begin{aligned} \int_L \mathcal{S} \rho(s^{-1} \cdot P_0) \psi(s) dm_G(s) &\geq \sum_{x \in P_0 \cap L_\delta} \left( \int_L \rho(s^{-1}x) dm_G(s) \right) \psi(x) \\ &\quad - \sum_{x \in P_0 \cap L_\delta} \int_L \rho(s^{-1}x) |\psi(s) - \psi(x)| dm_G(s) \\ &\geq \sum_{x \in P_0 \cap L_{-\delta}} \left( \int_L \rho(s^{-1}x) dm_G(s) \right) \psi(x) - \varepsilon |P_0 \cap L_\delta| \\ &= \sum_{x \in P_0 \cap L_{-\delta}} \psi(x) - \varepsilon |P_0 \cap L_\delta|, \end{aligned}$$

which is the desired lower bound.  $\square$

*Proof of Lemma 2.13.* Fix  $\delta > 0$  and choose a non-negative continuous function  $\rho$  on  $G$  supported on  $B_\delta$  with  $\int_G \rho dm_G = 1$ . We recall from Proposition 2.3 that  $M_\delta := \|\mathcal{S} \rho\|_\infty < \infty$ . Now let  $L \subset G$  be a compact set and note that for all  $x \in L_{-\delta}$ , we have  $L^{-1}x \supset B_\delta$ , and thus  $\int_L \rho(s^{-1}x) dm_G(s) = 1$ . We conclude that

$$\begin{aligned} M_\delta &\geq \frac{1}{m_G(L)} \int_L \mathcal{S} \rho(s^{-1} P_0) dm_G(s) = \frac{1}{m_G(L)} \sum_{x \in P_0} \int_L \rho(s^{-1}x) dm_G(s) \\ &\geq \frac{1}{m_G(L)} \sum_{x \in P_0 \cap L_{-\delta}} \int_L \rho(s^{-1}x) dm_G(s) = \frac{|P_0 \cap L_{-\delta}|}{m_G(L)}. \end{aligned}$$

$\square$

This completes the proof of Theorem 2.10.

*Proof of Corollary 2.11.* If  $G$  is amenable and  $X$  is compact and uniquely ergodic, then the point-wise ergodic theorem for uniquely ergodic systems (see e.g. [14, Thm. 4.10]) implies that  $C_c(G)$  is generic with respect to any Følner sequence. Then Corollary 2.11 follows from Theorem 2.10.  $\square$

**2.6. Gelfand pairs and spherical diffraction.** Following a suggestion by Hof, the *diffraction* of an FLC subset  $P_0 \subset \mathbb{R}^n$  is defined as the Fourier transform of the auto-correlation measure. This definition carries over to FLC sets in locally compact abelian groups. In generalizing diffraction theory to non-abelian lsc groups  $G$  we face the problem that there exist several different generalizations of the Fourier transform to non-abelian groups.

In this article, we focus on the case where  $G$  admits a compact subgroup  $K$ , such that  $(G, K)$  is a *Gelfand pair*. The basic theory of such pairs is summarized in Appendix B. From now on we use the notation of this appendix. In particular we denote by  $\mathcal{H}(G, K)$  the *Hecke algebra* of the pair  $(G, K)$  and by  $\mathcal{S}^+(G, K)$  the associated space of positive-definite spherical functions with its natural locally compact topology. In this setting, the *spherical Fourier transform* is defined as the transformation

$$\mathcal{F} : L^1(K \backslash G / K) \rightarrow C_0(\mathcal{S}^+(G, K)), \quad f \mapsto \hat{f}$$

given by

$$\hat{f}(\omega) = \int_G f(x) \omega(x^{-1}) dm_G(x) \quad (f \in L^1(K \backslash G / K), \omega \in \mathcal{S}^+(G, K)). \quad (2.5)$$

There exist several extensions of the spherical Fourier transform to various classes of generalized functions. The only one of relevance for the present article is the spherical Fourier transform of positive definite measures due to Selberg and Godement [17]. This transform associates with every positive definite Radon measure  $\mu$  on  $G$  a positive Radon measure  $\hat{\mu}$  on  $\mathcal{S}^+(G, K)$  in such a way that

$$\mu(f_1^* * f_2) = \int_{\mathcal{S}^+(G, K)} \overline{\hat{f}_1(\omega)} \cdot \hat{f}_2(\omega) d\hat{\mu}(\omega) \quad (f_1, f_2 \in \mathcal{H}(G, K)).$$

The measure  $\hat{\mu}$  is then called the *spherical Fourier transform* of  $\mu$ . In general it does not determine  $\mu$ , but it determines the *spherical part*  $\mu_{\text{sph}}$  of  $\mu$ , i.e. the restriction of  $\mu$  to the Hecke algebra.

Recall that the auto-correlation measure associated with an invariant probability measure  $\nu$  on the hull of an FLC set  $P_0 \subset G$  is positive definite. We may thus define:

**Definition 2.15.** Let  $(G, K)$  be a Gelfand pair,  $P_0 \subset G$  an FLC set with hull  $X$ , and assume that  $X$  admits a  $G$ -invariant probability measure  $\nu$ . Then the *spherical diffraction* of  $P_0$  with respect to  $K$  and  $\nu$  is the positive Radon measure

$$\hat{\eta}_\nu \in \mathcal{R}(\mathcal{S}^+(G, K)).$$

If  $X$  is uniquely ergodic and  $\nu$  is the unique invariant measure, then we simply refer to  $\hat{\eta} = \hat{\eta}_\nu$  as *the* spherical diffraction of  $P_0$ .

**Definition 2.16.** Let  $(G, K)$  be a Gelfand pair. We say that a subset  $P_0 \subset G$  of finite local complexity has *spherical pure point diffraction* if its hull is uniquely ergodic and there exists a countable subset  $\Omega \subset \mathcal{S}^+(G, K)$  and a function  $c \in \ell_{\text{loc}}^2(\Omega)$  such that the spherical diffraction  $\hat{\eta}$  of  $P_0$  is of the form

$$\hat{\eta} = \sum_{\omega \in \Omega} |c(\omega)|^2 \cdot \delta_\omega.$$

In the following we will construct many explicit examples of FLC sets with spherical pure point diffraction using cut-and-project schemes.

### 3. MODEL SETS AND THEIR HULLS

**3.1. General setting.** We now return to the setting described in the introduction. Throughout this section we assume that  $(G, H, \Gamma)$  is a model set triple and that  $W_0 \subset H$  is a  $\Gamma$ -regular window; we then denote by  $P_0 = P_0(G, H, \Gamma, W_0)$  the associated model set (see Definition 1.1). We denote by  $\pi_G, \pi_H$  the factor projections of the direct product  $G \times H$  and by  $\Gamma_G := \pi_G(\Gamma)$  and  $\Gamma_H := \pi_H(\Gamma)$  the projections of  $\Gamma$  onto  $G$  and  $H$  respectively. Using injectivity of the map  $\pi_G|_\Gamma : \Gamma \rightarrow \Gamma_G$  we define a map (which generalizes the classical “\*-map”)

$$\tau : \Gamma_G \rightarrow H, \quad g \mapsto \pi_H((\pi_G|_\Gamma)^{-1}(g)).$$

In terms of this map the set  $P_0$  can be described as

$$P_0 = \tau^{-1}(W_0). \tag{3.1}$$

By Lemma A.15, every model set is of finite local complexity, hence the results of the previous section apply to  $P_0$ . Following our previous convention we denote by  $X = X_{P_0}$  the hull of  $P_0$ . An important role in our study of the hull of a model set is played by the *canonical transversal*

$$\mathcal{T} := \{P \in X \mid P \subset \Gamma_G\}. \tag{3.2}$$

The name is justified by the fact that every  $G$ -orbit in  $X$  intersects  $\mathcal{T}$ , which is immediate from Proposition 2.2.(iv).

**Remark 3.1.** Our definition of a model set is less general than the most general versions which have been considered in the abelian case in that we always assume that the window is  $\Gamma$ -regular and that its boundary has zero Haar measure. Without the assumption of regularity of the window the construction of the parametrization map becomes substantially more technical. For windows with boundary of positive Haar measure the construction of the parametrization map works just fine, but this map will no longer be one-to-one over a set of full measure. In order to keep our presentation within reasonable size we leave these generalizations aside for now. On the other hand, our assumptions on the lattice  $\Gamma$  are very weak (and in particular we do not assume cocompactness).

**3.2. The parametrization map of a model set.** In this subsection we consider a model set  $P_0 = P_0(G, H, \Gamma, W_0)$  and relate its hull  $X$  to the *parameter space*

$$Y := (G \times H)/\Gamma.$$

We denote by  $y_0 := (e, e)\Gamma$  the basepoint of  $Y$ . The following theorem summarizes the main results of this subsection.

**Theorem 3.2** (Properties of the parametrization map). *There exists a unique  $G$ -equivariant Borel map  $\beta : X \rightarrow Y$  mapping which maps  $P_0$  to  $y_0$  and has a closed graph. This map has the following additional properties:*

- (i) If  $Y^{\text{ns}} := \{(g, h)\Gamma \in Y \mid h^{-1}W_0 \text{ is } \Gamma\text{-regular}\}$  and  $X^{\text{ns}} := \beta^{-1}(Y^{\text{ns}})$ , then

$$\beta|_{X^{\text{ns}}} : X^{\text{ns}} \rightarrow Y^{\text{ns}}$$

is bijective.

- (ii)  $P \in \mathcal{T}$  if and only if there exists  $h_P \in H$  such that  $\beta(P) = (e, h_P)\Gamma$ .  
 (iii) If  $P \in \mathcal{T} \cap X^{\text{ns}}$  and  $h_P$  is as in (ii), then  $P = \tau^{-1}(h_P^{-1}W_0)$ .  
 (iv) If  $\Gamma$  is cocompact, then  $\beta$  is continuous.

**Remark 3.3.** (1) In view of (i), elements of  $X^{\text{ns}}$  are called *non-singular points* and elements of  $Y^{\text{ns}}$  are called *non-singular parameters*.

- (2) The special case  $G = \mathbb{R}^k$ ,  $H = \mathbb{R}^n$  is classical. In this case,  $Y$  is a torus and  $\beta : X \rightarrow Y$  is known as the *torus parametrization* of the hull. For general locally compact abelian groups  $G$ ,  $H$  the construction of a parametrization map  $\beta$  is due to Schlottmann [49] (see also [48] for an earlier special case). In his proof he first establishes minimality of  $X$  using compactness (in the form of Gottschalk's criterion), and then uses minimality to establish existence of the parametrization map.  
 (3) Unlike Schlottmann's proof, the present argument does not require compactness of  $X$ , nor any a priori knowledge of minimality. Consequently, the argument also applies to non-compact hulls, and minimality comes for free in the case of compact hulls.  
 (4) Our proof of Theorem 3.2 does not use the full assumptions on  $\Gamma$  and  $W_0$ . We do not need that  $\Gamma$  is a lattice (as long as it is discrete and satisfies the other assumptions), nor do we use that  $W_0$  is Jordan-measurable. However, both assumptions will be used in the sequel. We need that  $W_0$  is Jordan-measurable to obtain that  $Y^{\text{ns}}$  has full Haar measure in  $Y$ , and that  $\Gamma$  is a lattice to obtain an invariant probability measure on  $X$ . We therefore pursue this additional generality here.

For the proof of Theorem 3.2 we first note that the assumptions on  $G$  imply that  $G$  is  $\sigma$ -compact. We may thus fix an exhaustion  $K_1 \subset K_2 \subset \dots \subset G$  of  $G$  by compact subsets. We also fix a sequence of symmetric pre-compact open identity neighbourhoods  $V_1 \supset V_2 \supset \dots$  in  $G$  such that  $\bigcap V_n = \{e\}$ . The proof is then based on the following two lemmas:

**Lemma 3.4.** *Let  $W_0 \subset H$  be a window, let  $h, h' \in H$  and let  $(h_n), (h'_n)$  be sequences in  $H$  converging to  $h$  and  $h'$  respectively.*

- (i) If  $h \neq h'$ , then there exists a non-empty, open subset  $U \subset H$  such that for all sufficiently large  $n \in \mathbb{N}$ ,

$$U \subset h_n W_0 \setminus h'_n W_0.$$

- (ii) If the windows  $h_n W_0$  and  $h W_0$  are  $\Gamma$ -regular, then for every  $K \in \mathcal{K}(G)$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have

$$(K \times h_n W_0) \cap \Gamma = (K \times h W_0) \cap \Gamma.$$

**Lemma 3.5.** *For every  $P \in \mathcal{T}$  there exists  $h_P \in H$  with the following property. For every sequence  $(g_n)$  in  $G$  with  $g_n P_0 \rightarrow P$  there exists a subsequence  $(g_{n_i})$  such that*

- (i)  $g_{n_i} = s_i \gamma_i$  for some  $s_i \in V_i$ ,  $\gamma_i \in \Gamma_G$ ;
- (ii)  $s_i \rightarrow e$ ,  $\tau(\gamma_i) \rightarrow h_P^{-1}$ ;
- (iii) For every  $i, j \in \mathbb{N}$  with  $j \geq i$  we have

$$\gamma_j P_0 \cap K_i = P \cap K_i.$$

Let us first explain how these lemmas imply the theorem.

*Proof of Theorem 3.2.* Consider the orbit closure  $Z := \overline{G \cdot (P_0, y_0)} \subset X \times Y$  and note that  $Z$  projects onto both  $X$  and  $Y$ . We claim that for every  $P \in X$  the section

$$Z[P] := \{y \in Y \mid (P, y) \in Z\} \tag{3.3}$$

is a singleton. Assuming the claim for the moment, we deduce that  $Z = \text{gr}(\beta)$  for some map  $\beta : X \rightarrow Y$ , which is  $G$ -equivariant by  $G$ -invariance of  $Z$  and satisfies  $\beta(P_0) = y_0$  by construction. Since  $\beta$  has a closed graph, it is automatically Borel. Conversely, if  $\beta' : X \rightarrow Y$  is any  $G$ -equivariant Borel map with closed graph satisfying  $\beta'(P_0) = y_0$ , then  $\text{gr}(\beta') \supset Z = \text{gr}(\beta)$  and thus  $\beta' = \beta$ . Thus our claim implies both existence and uniqueness of  $\beta$ . Moreover, in the cocompact case both  $X$  and  $Y$  are compact, hence  $\beta$  is automatically continuous by the closed graph theorem.

To establish the claim, consider first  $P \in \mathcal{T}$  and let  $y \in Z[P]$ . By definition this means that there exist  $g_n \in G$  such that

$$g_n \cdot (P_0, y_0) \rightarrow (P, y).$$

By Lemma 3.5 we can find a subsequence  $(g_{n_i})$  of  $(g_n)$  and  $s_i \in V_i$ ,  $\gamma_i \in \Gamma_G$  such that

$$y = \lim_{i \rightarrow \infty} g_{n_i} \cdot y_0 = \lim_{i \rightarrow \infty} (s_i \gamma_i, e) \Gamma = \lim_{i \rightarrow \infty} (s_i, \tau(\gamma_i)^{-1}) \Gamma = (e, h_P) \Gamma.$$

Thus  $Z[P] = \{(e, h_P) \Gamma\}$  is a singleton.

Now let  $P \in X$  be arbitrary. Since  $P \neq \emptyset$  we can pick  $p \in P$ . By Proposition 2.2.(iv) we have  $p^{-1}P \in \mathcal{T}$ , hence if  $\{y_1, y_2\} \in Z[P]$ , then by  $G$ -invariance of  $Z$  we have

$$\{p^{-1}y_1, p^{-1}y_2\} \subset Z[p^{-1}P],$$

and thus  $p^{-1}y_1 = p^{-1}y_2$  by the previous argument. This implies  $y_1 = y_2$  and finishes the proof of the claim and shows that  $Z = \text{gr}(\beta)$ . For  $P \in \mathcal{T}$  we have also established that

$$\beta(P) = (e, h_P) \Gamma. \tag{3.4}$$

To show (iii) we consider  $P \in X^{\text{ns}} \cap \mathcal{T}$  and, using Lemma 3.5, pick a sequence  $(\gamma_i)$  in  $\Gamma_G$  such that  $\gamma_i P_0 \rightarrow P$ ,  $\tau(\gamma_i) \rightarrow h_P^{-1}$  and for all  $j \geq i$

$$P \cap K_i = \gamma_j P_0 \cap K_i. \tag{3.5}$$

Now fix  $i \in \mathbb{N}$  and consider the finite sets  $F := (K_i \times h_P^{-1} W_0) \cap \Gamma$  and  $F_j := (K_i \times \tau(\gamma_j) W_0) \cap \Gamma$ . Since  $P \in X^{\text{ns}} \cap \mathcal{T}$  and  $\beta(P) = (e, h_P) \Gamma$ , the window  $h_P^{-1} W_0$  is  $\Gamma$ -regular, and every  $\Gamma_H$ -translate of  $W_0$



is regular as well. Since  $\tau(\gamma_j) \rightarrow h_p^{-1}$  we can thus apply Lemma 3.4.(ii) to find  $j \geq i$  such that  $F = F_j$ . For such  $j$  we can then apply (3.5) to obtain

$$P \cap K_i = \gamma_j P_0 \cap K_i = \pi_G(F_j) = \pi_G(F) = \tau^{-1}(h_p^{-1} W_0) \cap K_i.$$

Since  $i$  was arbitrary, this implies  $P = \tau^{-1}(h_p^{-1} W_0)$ . This finishes the proof of (iii) and shows that  $\beta$  is injective on  $X^{ns} \cap \mathcal{T}$ .

We now establish (ii). The inclusion  $\mathcal{T} \subset \beta^{-1}(\{e\} \times H)\Gamma$  has already been established in (3.4). Conversely assume that  $P \in X$  with  $\beta(P) = (e, h)\Gamma$  for some  $h \in H$ . By Proposition 2.2.(iii) we have  $p^{-1}P \in \mathcal{T}$  for every  $p \in P$ , hence there exists  $h_p \in H$  such that  $\beta(p^{-1}P) = (e, h_p)\Gamma$ . It then follows from  $G$ -equivariance of  $\beta$  that

$$(e, h_p)\Gamma = \beta(p^{-1}P) = p^{-1}\beta(P) = (p^{-1}, h)\Gamma,$$

hence  $p \in \Gamma_G$ . Since  $p \in P$  was arbitrary this implies  $P \in \mathcal{T}$  and finishes the proof of (ii).

Concerning (i), assume that  $P_1, P_2 \in X^{ns}$  satisfy  $\beta(P_1) = \beta(P_2) = (g, h)\Gamma$  for some  $g \in G, h \in H$ . Then

$$\beta(g^{-1}P_1) = \beta(g^{-1}P_2) = (e, h)\Gamma,$$

and hence  $\{g^{-1}P_1, g^{-1}P_2\} \subset X^{ns} \cap \mathcal{T}$  by (ii). Since  $\beta$  is injective on  $X^{ns} \cap \mathcal{T}$  we deduce that  $g^{-1}P_1 = g^{-1}P_2$  and hence  $P_1 = P_2$ . This proves (i) and finishes the proof.  $\square$

It remains to prove the lemmas.

*Proof of Lemma 3.4.* (i) Given  $n \in \mathbb{N}$  we define a subset  $W_{0,n} \subset W_0^o$  by

$$W_{0,n} := \{w \in W_0^o \mid V_n^{-1}w \subseteq W_0^o\} \quad (n \in \mathbb{N}).$$

We first observe that

$$W_{0,1} \subset W_{0,2} \subset \dots \subset W_0^o \quad \text{and} \quad W_{0,n} \subset \bigcap_{v \in V_n} vW_0. \quad (3.6)$$

Indeed, the first statement follows from the fact that the  $V_n$  are decreasing, and if  $v \in V_n$  and  $w \in W_{0,n}$  then  $v^{-1}w \in W_0^o \subseteq W_0$ , which implies  $w \in vW_0$  and thus establishes (3.6).

Now define  $M_n := hW_{0,n} \setminus h'\overline{V_n}W_0$ . We claim that there exists  $n \in \mathbb{N}$  such that  $M_n$  contains a non-empty open set  $U$ . Assuming the claim for the moment, let us finish the proof. We can find  $k_0$  such that for all  $k \geq k_0$  we have  $h_k \in hV_n$  and  $h'_k \in h'V_n$ . Then by (3.6) we have

$$h_k W_0 \setminus h'_k W_0 \supset \bigcap_{v \in V_n} h v W_0 \setminus h' V_n W_0 \supset h \bigcap_{v \in V_n} v W_0 \setminus h' \overline{V_n} W_0 \supset M_n \supset U,$$

i.e.  $h_k W_0 \setminus h'_k W_0 \supset U$ , which is the statement of the lemma. It thus remains to establish the claim.

Firstly, since  $W_0$  has trivial stabilizer we have  $hW_0 \setminus h'W_0 \neq \emptyset$ . Secondly, since  $h'W_0$  is compact and  $hW_0^o$  is dense in  $hW_0$ ,  $hW_0^o \setminus h'W_0 \subset hW_0 \setminus h'W_0$  is dense, and in particular  $hW_0^o \setminus h'W_0 \neq \emptyset$ . Thirdly,  $hW_0^o \setminus h'W_0$  is open, and thus  $m_H(hW_0^o \setminus h'W_0) > 0$ . From regularity of the Haar measure we thus deduce that

$$\exists m \in \mathbb{N} : m_H(hW_0^o \setminus h'\overline{V_m}W_0) > 0. \quad (3.7)$$

We fix such an  $m$  once and for all and observe that  $A := hW_0^o \setminus h'\overline{V_m}W_0 \neq \emptyset$ .

Since the set  $A$  is open and non-empty it contains a basic open set of the form  $U_{n,w} := hV_n^{-1}h^{-1}V_n w$  for some  $n \in \mathbb{N}$  and  $w \in G$ . We may assume that  $n \geq m$ . Then we claim that

$$U := V_n w \subset M_n. \quad (3.8)$$

Since  $U$  is open this will finish the proof. From the inclusion  $U_{n,w} \subset A$  we deduce two things: Firstly,  $U_{n,w} \subset hW_0^o$ , i.e.  $V_n^{-1}h^{-1}V_n w \subset W_0^o$ . By the very definition of  $W_{0,n}$  this means that  $h^{-1}V_n w \subset W_{0,n}$ , and hence

$$U = V_n w \subset hW_{0,n}. \quad (3.9)$$

Secondly, since  $e \in V_n^{-1}$  we have  $U = V_n w \subset U_{n,w}$  and thus

$$U \cap h' \overline{V_m} W_0 \subset U_{n,w} \cap h' \overline{V_m} W_0 \subset A \cap h' \overline{V_m} W_0 = \emptyset. \quad (3.10)$$

Combining (3.10) and (3.9) and using that  $n \geq m$  and hence  $\overline{V_n} \subset \overline{V_m}$  we obtain

$$U \subset h W_{0,n} \setminus h' \overline{V_m} W_0 \subset h W_{0,n} \setminus h' \overline{V_n} W_0 = M_n.$$

This establishes (3.8) and finishes the proof.

(ii) Let  $K \subset G$  be a compact set. Since  $h W_0$  and  $h_n W_0$  are  $\Gamma$ -regular for every  $n$ , we have

$$(K \times h W_0) \cap \Gamma = ((K \times (h W_0 \setminus h_n W_0^o)) \cap \Gamma) \cup ((K \times (h W_0 \cap h_n W_0)) \cap \Gamma) \quad (3.11)$$

and

$$(K \times h_n W_0) \cap \Gamma = ((K \times (h_n W_0 \setminus h W_0^o)) \cap \Gamma) \cup ((K \times (h_n W_0 \cap h W_0)) \cap \Gamma). \quad (3.12)$$

Hence, in order to show that

$$(K \times h W_0) \cap \Gamma = (K \times h_n W_0) \cap \Gamma \quad (3.13)$$

for large enough  $n$ , it suffices to show that for large enough  $n$ , we have

$$(K \times (h W_0 \setminus h_n W_0^o)) \cap \Gamma = (K \times (h_n W_0 \setminus h W_0^o)) \cap \Gamma = \emptyset.$$

Since  $W_0$  is compact and  $h_n \rightarrow h$ , there is a compact set  $L \subset H$  such that  $h W_0 \setminus h_n W_0^o \subset L$  and  $h_n W_0 \setminus h W_0^o \subset L$  for all  $n$ . Since  $\Gamma$  is discrete, this shows that the sets

$$A_n = (K \times (h W_0 \setminus h_n W_0^o)) \cap \Gamma \quad \text{and} \quad B_n = (K \times (h_n W_0 \setminus h W_0^o)) \cap \Gamma$$

vary inside the set of all subsets of the *finite* set  $T = (K \times L) \cap \Gamma$ . In particular, for every sub-sequence of  $(h_n)$ , there is a further sub-sequence  $(h_{n_j})$  such that the sequences  $(A_{n_j})$  and  $(B_{n_j})$  are constant. On the other hand, one readily verifies that

$$\bigcap_j (h W_0 \setminus h_{n_j} W_0^o) \subset h \partial W_0 \quad \text{and} \quad \bigcap_j (h_{n_j} W_0 \setminus h W_0^o) \subset h \partial W_0,$$

for every sub-sequence  $(h_{n_j})$ , and thus, since  $h W_0$  is  $\Gamma$ -regular, we conclude that

$$\bigcap_j A_{n_j} = \emptyset \quad \text{and} \quad \bigcap_j B_{n_j} = \emptyset.$$

We conclude that every sub-sequence of  $(h_n)$  admits a further sub-sequence  $(h_{n_j})$  such that

$$(K \times (h W_0 \setminus h_{n_j} W_0^o)) \cap \Gamma = (K \times (h_{n_j} W_0 \setminus h W_0^o)) \cap \Gamma = \emptyset.$$

for all  $j$ . We claim that this finishes the proof. Indeed, if (3.13) were to fail for infinitely many  $n$ , then (3.11) and (3.12) would tell us that we can find a sub-sequence  $(h_{n_j})$  such that either

$$(K \times (h W_0 \setminus h_{n_j} W_0^o)) \cap \Gamma \neq \emptyset \quad \text{or} \quad (K \times (h_{n_j} W_0 \setminus h W_0^o)) \cap \Gamma \neq \emptyset,$$

for all  $j$ , which contradicts what we have just proved.  $\square$

*Proof of Lemma 3.5.* Since  $g_n P_0 \rightarrow P$  we can find for every  $i \in \mathbb{N}$  some  $n_i \in \mathbb{N}$  and  $t_i \in V_i$  such that

$$t_i g_{n_i} P_0 \cap K_i = P \cap K_i. \quad (3.14)$$

If we define  $\gamma_i := t_i g_{n_i}$  then  $\gamma_i P_0 \cap K_i = P \cap K_i$ . We deduce that for all  $j \geq i$  we have

$$\gamma_j P_0 \cap K_i = (\gamma_j P_0 \cap K_j) \cap K_i = P \cap K_j \cap K_i = P \cap K_i.$$

Since  $P \cap K_i \neq \emptyset$ , we may assume by passing to a further subsequence that  $\gamma_i P_0 \cap K_i \neq \emptyset$  for all  $i \in \mathbb{N}$ . Then for every  $i \in \mathbb{N}$  there exists  $p_0 \in P_0 \subset \Gamma_G$  such that  $\gamma_i p_0 \in \gamma_i P_0 \cap K_i = P \cap K_i$ . Since  $P \in \mathcal{T}$  we deduce that  $\gamma_i p_0 \in \Gamma_G$  and thus  $\gamma_i \in \Gamma_G$ . If we now set  $s_i := t_i^{-1}$ , then  $s_i \in V_i^{-1} = V_i$  and  $g_{n_i} = s_i \gamma_i$ . For this choice of  $\gamma_i$  and  $s_i$  we have thus established (i) and (iii). Also note that  $s_i \in V_i$  implies  $s_i \rightarrow e$ .

We next claim that the set  $\{\tau(\gamma_i)\}$  is pre-compact. Suppose otherwise for contradiction. Then for every  $i \in \mathbb{N}$  there exists  $j > i$  such that

$$\tau(\gamma_i)W_0 \cap \tau(\gamma_j)W_0 = \emptyset,$$

and consequently

$$\pi_G([(K_i \times \tau(\gamma_i)W_0) \cap \Gamma] \cap [(K_j \times \tau(\gamma_j)W_0) \cap \Gamma]) = \emptyset.$$

Since  $\pi_G|_\Gamma$  is injective, this can be rewritten as

$$\pi_G(K_i \times \tau(\gamma_i)W_0) \cap \Gamma \cap \pi_G(K_j \times \tau(\gamma_j)W_0) \cap \Gamma = \emptyset,$$

or equivalently

$$(\gamma_i P_0 \cap K_i) \cap (\gamma_j P_0 \cap K_j) = \emptyset.$$

This, however, contradicts (iii), and establishes our claim.

In order to establish (ii) and thereby to finish the proof of the lemma it remains to show that every convergent subsequence of  $\tau(\gamma_i)$  converges to the same limit  $h_p^{-1}$ , which is independent of the sequence  $g_n$ . We argue again by contradiction and assume otherwise. Then there exist cofinal subsets  $I, I' \subset \mathbb{N}$  and sequences  $(\gamma_i)_{i \in I}, (\gamma'_{i'})_{i' \in I'}$  satisfying (iii) such that  $h_i := \tau(\gamma_i)$  and  $h'_{i'} := \tau(\gamma'_{i'})$  converge to different elements of  $H$ . By Lemma 3.4.(i) we then find an open set  $U$  such that for all sufficiently large  $i, i'$ ,

$$U \subset \tau(\gamma_i)W_0 \setminus \tau(\gamma'_{i'})W_0.$$

Since  $\Gamma_H$  is dense in  $H$  we have  $\Gamma_H \cap U \neq \emptyset$  and thus  $(G \times U) \cap \Gamma \neq \emptyset$ . For sufficiently large  $j$  we thus get  $(K_j \times U) \cap \Gamma \neq \emptyset$ , hence

$$\emptyset \neq (K_j \times U) \cap \Gamma \subset (K_j \times \tau(\gamma_i)W_0) \cap \Gamma \setminus [(K_j \times \tau(\gamma'_{i'})W_0) \cap \Gamma].$$

Applying  $\pi_G$ , which is injective on  $\Gamma$ , we obtain

$$\emptyset \neq (\gamma_i P_0 \cap K_j) \setminus (\gamma'_{i'} P_0 \cap K_j).$$

However, we may assume that  $i$  and  $i'$  are larger than  $j$ . Then (iii) yields

$$\gamma_i P_0 \cap K_j = P \cap K_j = \gamma'_{i'} P_0 \cap K_j,$$

which is a contradiction.  $\square$

**3.3. Minimality and unique ergodicity of the hull.** Using the parametrization map it is now straight-forward to establish the desired minimality and unique ergodicity properties of the hull. We recall that a subset  $P \subset G$  is called *left-syndetic* if there exists a compact subset  $K \subset G$  such that  $G = PK$ . By Proposition A.19 the model set  $P_0$  is left-syndetic if  $\Gamma$  is cocompact in  $G \times H$ .

**Proposition 3.6** (Minimality properties).

- (i)  $Y$  is a minimal  $G$ -space.
- (ii) If  $P_0$  is left-syndetic, then  $X$  is a minimal compact  $G$ -space.
- (iii) If  $P_0$  is not left-syndetic, then  $X$  is not compact and no  $G$ -orbit in  $X$  is pre-compact.

*Proof.* Observe first that since  $\Gamma_H$  is dense,  $G \setminus (G \times H)$  is minimal as a  $\Gamma$ -space, and thus  $Y = (G \times H)/\Gamma$  is minimal as a  $G$ -space by the duality principle. Now assume that  $X_0 \subset X$  is a non-empty compact  $G$ -invariant subset. Since  $\beta$  has a closed graph, it maps compact sets to closed sets, and since it is  $G$ -equivariant, the image of  $X_0$  is a closed  $G$ -invariant subset of  $Y$ . Minimality of  $Y$  then yields  $\beta(X_0) = Y$ . In particular  $\beta(P_0) \in \beta(X_0)$ , and since  $P_0 \in X^{\text{ns}}$  we deduce  $P_0 \in X_0$  and thus  $X_0 = X$ . This proves that every compact  $G$ -invariant subset of  $X$  is either empty or all of  $X$ . In view of Corollary A.18 this implies (ii) and (iii).  $\square$

We now turn to the question of unique ergodicity of the hull. It is an immediate consequence of the duality principle that  $(Y, m_Y)$  is (uniquely)  $G$ -ergodic, and we want to lift this property to  $X$ . The natural context to discuss this problem is that of stationary measures. Recall that a probability measure  $\mu$  on  $G$  is called *admissible* if its support generates  $G$  and if it is absolutely continuous with respect to the Haar measure  $m_G$  on  $G$ . Then a probability measure  $\nu$  on a measurable  $G$ -space is called  $\mu$ -stationary provided  $\mu * \nu = \nu$ . If  $G$  is a non-amenable group and  $Z$  is a compact  $G$ -space, then there might not exist  $G$ -invariant measures, but there will always exist  $\mu$ -stationary measures for any admissible probability measure  $\mu$  on  $G$ . The following theorem classifies stationary measures on  $X$  and implies in particular unique ergodicity of the hull.

**Theorem 3.7** (Classification of stationary measures on  $X$ ). *There exists a unique  $G$ -invariant probability measure  $\nu$  on  $X$ . This measure satisfies  $\nu(X^{\text{ns}}) = 1$  and is also the unique stationary probability measure with respect to any admissible probability measure  $\mu$  on  $G$ .*

Moreover,  $\beta : (X, \nu) \rightarrow (Y, m_Y)$  is a measurable isomorphism of  $G$ -spaces and thus induces an isomorphism

$$\beta^* : L^2(Y, m_Y) \rightarrow L^2(X, \nu)$$

of unitary  $G$ -representations.

The proof of Theorem 3.7 is a consequence of the following two lemmas.

**Lemma 3.8.** *The subset  $Y^{\text{ns}} \subset Y$  is conull with respect to Haar measure on  $Y$ .*

**Lemma 3.9.** *The Haar measure  $m_Y$  is the unique stationary probability measure on  $Y$  with respect to any admissible probability measure  $\mu$  on  $G$ .*

*Proof of Theorem 3.7.* Let  $\beta : X \rightarrow Y$  be the parametrization map constructed in Theorem 3.2. Since  $\beta$  has a closed graph, so does the restriction  $\beta|_{X^{\text{ns}}} : X^{\text{ns}} \rightarrow Y^{\text{ns}}$  and hence also its inverse  $(\beta|_{X^{\text{ns}}})^{-1} : Y^{\text{ns}} \rightarrow X^{\text{ns}}$ . In particular,  $(\beta|_{X^{\text{ns}}})^{-1}$  is Borel and we may define a  $G$ -invariant probability measure on  $X$  by

$$\nu := (\beta|_{X^{\text{ns}}})_*^{-1} m_Y|_{Y^{\text{ns}}}.$$

By definition, we have for every Borel subset  $A \subset X$ ,

$$\nu(A) = m_Y(\beta(A \cap X^{\text{ns}})) = m_Y(\beta(A) \cap Y^{\text{ns}}) = m_Y(\beta(A)).$$

Now let  $\mu$  be an admissible probability measure on  $G$  and  $\nu'$  a  $\mu$ -stationary probability measure on  $X$ . Then  $\beta_* \nu'$  is a  $\mu$ -stationary measure on  $Y$  and thus  $\beta_* \nu' = m_Y$  by Lemma 3.9. Since  $\beta_* \nu'(Y^{\text{ns}}) = 1$  by Lemma 3.8 we deduce that  $\nu'(X^{\text{ns}}) = 1$ , i.e.  $\nu'$  is a probability measure on  $X^{\text{ns}}$ . Now  $\mu$ -stationary measures on  $X^{\text{ns}}$  correspond bijectively via  $\beta$  to  $\mu$ -stationary measures on  $Y^{\text{ns}}$  via the  $G$ -equivariant Borel isomorphism  $\beta|_{X^{\text{ns}}}$ . We conclude that  $\nu' = \nu$ , and the theorem follows.  $\square$

*Proof of Lemma 3.8.* An element  $(g, h) \cdot \Gamma \in Y$  is singular if and only if  $h^{-1}W_0$  is not  $\Gamma$ -regular. This amounts to  $h^{-1}\partial W_0 \cap \Gamma_H \neq \emptyset$ , i.e.  $h \in \partial W_0 \Gamma_H$ . Now since  $W_0$  is Jordan-measurable we have  $m_H(\partial W_0) = 0$  and thus  $\partial W_0 \Gamma_H$  is a nullset by countability of  $\Gamma_H$ .  $\square$

*Proof of Lemma 3.9.* Fix an admissible probability measure  $\mu$  on  $G$  and let  $\nu$  be an arbitrary  $\mu$ -stationary probability measure on  $Y$ . We are going to show that  $\nu = m_Y$ .

For every non-negative function  $\rho \in C_c(H)$  normalized to  $\int \rho dm_H = 1$  we define a probability measure  $\nu_\rho$  on  $Y$  by  $\nu_\rho := (\mu \otimes \rho m_H) * (\mu \otimes \rho m_H) * \nu$ . Using that the  $G$ - and  $H$ -action commute, we see that  $\nu_\rho$  is  $\mu$ -stationary. Since  $\mu$  and  $\rho m_H$  are respectively absolutely continuous with respect to  $m_G$  and  $m_H$  we deduce that  $(\mu \otimes \rho m_H) * \nu$  is absolutely continuous with respect to  $m_Y$ . The second convolution then has a smoothing effect, and we deduce that  $\nu_\rho$  has a continuous density  $\psi_\rho \in C(Y)$  with respect to Haar measure. Since  $m_Y$  is  $G$ -invariant and  $\nu_\rho$  is  $\mu$ -stationary, the density  $\psi_\rho$  is  $\mu$ -stationary as well. By a standard argument, this implies that  $\psi_\rho$  is actually

$G$ -invariant. Indeed, since  $\psi_\rho$  is continuous and the support of  $\mu$  generates  $G$  as a semigroup, it suffices to show that for all  $k \in \mathbb{N}$ ,

$$I_k := \int_G \int_Y (\psi_\rho(gy) - \psi_\rho(y))^2 dm_Y(y) d\mu^{*k}(g) = 0,$$

Using stationarity of  $m_Y$  and  $\psi_\rho$  and expanding the square we obtain

$$I_k = 2 \left( \int_Y \psi_\rho^2 dm_Y - \int_Y \psi_\rho(y) \left[ \int \psi_\rho(gy) d\mu^{*k}(g) \right] dm_Y(y) \right) = 0$$

for all  $k \in \mathbb{N}$ . This shows that  $\psi_\rho$  is indeed  $G$ -invariant, hence constant by Proposition 3.6.(i). We deduce that  $\psi_\rho = 1$  and  $\nu_\rho = m_H$  for every  $\rho$  as above.

Now let  $\rho_n$  be a sequence of normalized positive functions in  $C_c(H)$  such that  $\rho_n m_H$  converges to  $\delta_e$  in the weak-\* topology. Then the previous argument yields  $\nu_{\rho_n} = m_Y$  for every  $n$  and thus

$$\nu = \lim_{n \rightarrow \infty} \nu_{\rho_n} = m_Y. \quad \square$$

#### 4. AUTO-CORRELATION AND SPHERICAL DIFFRACTION OF MODEL SETS

**4.1. A formula for the auto-correlation of a model set.** Throughout this section we fix a model set  $P_0 = P_0(G, H, \Gamma, W_0)$ . We denote by  $X$  the hull of  $P_0$  and by  $\nu$  the unique  $G$ -invariant probability measure on  $X$ . Finally, we set  $Y := (G \times H)/\Gamma$  and denote by  $\beta : X \rightarrow Y$  the parametrization map. We normalize Haar measures  $m_G$  and  $m_H$  on  $G$  and  $H$  in such a way that  $\text{covol}(\Gamma) = 1$  and hence  $\beta_* \nu = m_Y$ .

Given a bounded Riemann-integrable function  $F : G \times H \rightarrow \mathbb{R}$  with compact support we denote by  $\mathcal{P}_\Gamma F$  the  $\Gamma$ -periodization of  $F$ , i.e. the function

$$\mathcal{P}_\Gamma F : Y \rightarrow \mathbb{R}, \quad \mathcal{P}_\Gamma F((g, h)\Gamma) = \sum_{\gamma \in \Gamma} F((g, h)\gamma).$$

**Theorem 4.1** (Auto-correlation of model sets). *Let  $\eta = \eta_\nu$  be the auto-correlation measure of  $P_0$ . Then for every  $f \in C_c(G)$ ,*

$$\eta(f^* * f) = \|\mathcal{P}_\Gamma(f \otimes \chi_{W_0})\|_{L^2(Y)}^2. \quad (4.1)$$

In view of (2.1) this is an immediate consequence of the following lemma and the fact that  $\nu(X^{\text{ns}}) = m_Y(Y^{\text{ns}}) = 1$ .

**Lemma 4.2.** *If  $P \in X^{\text{ns}}$  and  $f \in C_c(G)$ , then*

$$\mathcal{S}f(P) = \mathcal{P}_\Gamma(f \otimes \chi_{W_0})(\beta(P)).$$

*Proof.* Let  $P \in X^{\text{ns}}$  and  $p \in P$ . By Proposition 2.2.(iv) and  $G$ -invariance of  $X^{\text{ns}}$  we then have  $P' := p^{-1}P \in \mathcal{T} \cap X^{\text{ns}}$ . By Theorem 3.2 we then have

$$\beta(P') = (e, h_{P'})\Gamma \quad \text{and} \quad \tau^{-1}(h_{P'}^{-1}W_0) = P'.$$

Now  $G$ -equivariance of  $\beta$  yields

$$\beta(P) = \beta(pP') = p\beta(P') = p(e, h_{P'})\Gamma = (p, h_{P'})\Gamma,$$

and thus we obtain for every  $f \in C_c(G)$ ,

$$\begin{aligned} \mathcal{P}_\Gamma(f \otimes \chi_{W_0})(\beta(P)) &= \mathcal{P}_\Gamma(f \otimes \chi_{W_0})(p, h_{P'})\Gamma = \sum_{\gamma \in \Gamma_G} f(p\gamma) \chi_{W_0}(h_{P'}\tau(\gamma)) \\ &= \sum_{\gamma \in \tau^{-1}(h_{P'}^{-1}W_0)} f(p\gamma) = \sum_{\gamma \in P'} f(p\gamma) = \sum_{\gamma \in pP'} f(\gamma) = \sum_{\gamma \in P} f(\gamma) \\ &= \mathcal{S}f(P). \end{aligned}$$

$\square$

We mention in passing that Lemma 4.2 also yields the Siegel constant of  $P_0$ :

**Corollary 4.3** (Siegel constants of model sets). *If the Haar measure on  $G$  and  $H$  are normalized as above and  $\nu$  denotes the unique invariant measure on  $X$ , then*

$$h_{P_0, \nu, m_G} = m_H(W_0).$$

*Proof.* Since  $\beta_* \nu = m_Y$  and  $\text{covol}(\Gamma) = 1$ , Lemma 4.2 implies that for every  $f \in C_c(G)$ ,

$$\begin{aligned} \int_X \mathcal{S}f d\nu &= \int_Y \mathcal{P}(f \otimes \chi_{W_0}) dm_Y \\ &= \text{covol}(\Gamma) \cdot \int_G \int_H f \otimes \chi_{W_0} dm_G dm_H \\ &= m_H(W_0) \cdot m_G(f). \end{aligned}$$

□

Combining this observation with Remark 2.7 yields:

**Corollary 4.4.** *Let  $\rho \in C_c(G)$  with  $\rho = \rho^*$ ,  $\rho \geq 0$  and  $\int \rho dm_G = 1$  and assume that  $\text{supp}(\rho)$  generates  $G$ . Then*

$$m_H(W_0) = \sqrt{\lim_{n \rightarrow \infty} \eta(\rho^{*n})}.$$

□

**4.2. Cocompact lattices and the shadow transform.** Our next goal is to study the spherical diffraction of model sets  $P_0 = P_0(G, H, \Gamma, W_0)$ . For this to be defined we need to assume that there exists a compact subgroup  $K < G$  such that  $(G, K)$  is a Gelfand pair. We fix such a Gelfand pair from now on, and denote by  $\mathcal{H}(G, K)$  and  $\mathcal{S}^+(G, K)$  the Hecke algebra, respectively the space of positive-definite spherical functions of the pair  $(G, K)$ . Given a function  $f \in L^1(K \backslash G / K)$  or a positive definite Radon measure  $\mu$  on  $G$  we denote by  $\hat{f}$  respectively  $\hat{\mu}$  the corresponding spherical Fourier transforms (see Subsection 2.6 and Appendix B).

For this and the following subsection we shall also assume that  $\Gamma$  is a *cocompact* lattice in  $G \times H$ . As before we abbreviate  $Y := (G \times H) / \Gamma$ . We also denote the inner product of  $L^2(Y)$  simply by  $\langle -, - \rangle$ . The  $G$ -action on  $Y$  induces a unitary representation on  $L^2(Y)$  and thus an action of the Hecke algebra  $\mathcal{H}(G, K)$  on  $L^2(Y)^K$ . Given  $\omega \in \mathcal{S}^+(G, K)$  we denote by

$$L^2(Y)_\omega^K = \{g \in L^2(Y)^K \mid \forall f \in \mathcal{H}(G, K): f * g = \hat{f}(\omega) \cdot g\}$$

the associated weight space. Then the *spherical spectrum* of  $Y$  is the subset

$$\Omega_Y := \{\omega \in \mathcal{S}^+(G, K) \mid L^2(Y)_\omega^K \neq \{0\}\} \subset \mathcal{S}^+(G, K).$$

**Proposition 4.5** (Spectral decomposition). *Let  $\Gamma$  be a cocompact lattice in  $G \times H$ .*

- (i) *The  $G$ -representation  $L^2(Y)$  is completely reducible with countable multiplicities.*
- (ii) *The spherical spectrum  $\Omega_Y \subset \mathcal{S}^+(G, K)$  is a countable subset of  $\mathcal{S}^+(G, K)$  and the  $\mathcal{H}(G, K)$ -representation  $L^2(Y)^K$  decomposes as*

$$L^2(Y)^K = \widehat{\bigoplus_{\omega \in \Omega_Y} L^2(Y)_\omega^K}. \quad (4.2)$$

*Proof.* (i) Since  $\Gamma$  is cocompact in  $G \times H$ , the  $(G \times H)$ -representation  $L^2(Y)$  is completely reducible with finite multiplicities (see e.g. [55, Thm. 7.2.5]). Since  $(G, K)$  is a Gelfand pair, the group  $G$  is of type I (see e.g. [11, Thm. 2.2]). Consequently, every irreducible unitary representation  $(G \times H)$ -representation is of the form  $V \boxtimes W$  where  $V$  is an irreducible unitary  $G$ -representation,  $W$  is an irreducible unitary  $H$ -representation and  $V \boxtimes W$  is isomorphic to the completed tensor

product of  $V$  and  $W$  with  $(G \times H)$ -action given by  $(g, h).(v \otimes w) = gv \otimes hw$  (see e.g. [16, Thm. 7.25]). In this situation, if  $(w_i)_{i \in I}$  is a Hilbert space basis of  $W$  then, as  $G$ -representations,

$$V \boxtimes W|_G \cong \bigoplus_{i \in I} V \otimes \mathbb{C} \cdot w_i \cong \bigoplus_{i \in I} V.$$

Note that  $I$  is countable, since  $L^2(Y)$  and hence  $W$  are separable. We deduce that, as  $G$ -representations, each  $V \boxtimes W$  and thus also  $L^2(Y)$  are completely reducible with countable multiplicities.

(ii) follows from (i) and Lemma B.5 in Appendix B.  $\square$

The decomposition (4.2) is called the *spectral decomposition* of  $L^2(Y)$ . Note that cocompactness of  $\Gamma$  is essential for discreteness of the spectrum. Given  $\omega \in \Omega_Y$  we write

$$\pi_\omega : L^2(Y)^K \rightarrow L^2(Y)_\omega^K$$

for the orthogonal projection. We also write  $\langle -, - \rangle_\omega$  for the inner product in  $L^2(Y)_\omega^K$  and  $\| \cdot \|_\omega$  for the corresponding norm.

**Proposition 4.6** (Construction of the shadow transform). *Let  $r \in C_c(H)$  and  $\omega \in \Omega_Y$ . Then there exists a unique  $\hat{r}(\omega) \in L^2(Y)_\omega^K$  such that*

$$\langle \mathcal{P}_\Gamma(f \otimes r), \psi \rangle = \overline{\hat{r}(\omega)} \cdot \langle \hat{r}(\omega), \psi \rangle_\omega \quad \text{for all } f \in \mathcal{H}(G, K) \text{ and } \psi \in L^2(Y)_\omega^K.$$

**Definition 4.7.** The map  $\mathbb{S} : C_c(H) \rightarrow \text{Map}(\Omega_Y, L^2(Y))$  given by  $\mathbb{S}(r) := \hat{r}$  is called the *shadow transform* of the cocompact lattice  $\Gamma$ .

The following corollaries are immediate consequences.

**Corollary 4.8.** *Let  $r \in C_c(H)$ ,  $\omega \in \Omega_Y$  and  $f \in \mathcal{H}(G, K)$  with  $\hat{f}(\omega) = 1$ . Then*

$$\hat{r}(\omega) = \pi_\omega(\mathcal{P}_\Gamma(f \otimes r)). \quad \square$$

**Corollary 4.9.** *Let  $r \in C_c(H)$ . Then the following are equivalent:*

- (i)  *$r$  is in the kernel of the shadow transform, i.e.  $\hat{r}(\omega) = 0$  for all  $\omega \in \Omega_Y$ .*
- (ii)  *$\mathcal{P}_\Gamma(f \otimes r) = 0$  for all  $f \in \mathcal{H}(G, K)$ .*

$\square$

*Proof of Proposition 4.6.* Let  $\psi \in L^2(Y)_\omega^K$  and let  $\tilde{\psi}$  be the lift of  $\psi$  to a right  $\Gamma$ -invariant bounded function class on  $G \times H$ . We define

$$\phi(g, h) := \int_K \tilde{\psi}(gk, h) dm_K(k),$$

where  $m_K$  denotes the Haar probability measure on  $K$ . Since  $\psi$  is an eigenfunction of  $\mathcal{H}(G, K)$ ,  $\tilde{\psi}$  is continuous in the  $G$ -variable. We can thus find a Borel representative  $\tilde{\psi}_0$  of the class  $\tilde{\psi}$ , which is continuous in the first variable. Using such a representative we then define for every  $h \in H$  a function  $\phi_h : G \rightarrow \mathbb{C}$  by

$$\phi_h(g) := \int_K \tilde{\psi}_0(gk, h) dm_K(k).$$

The functions  $\phi_h$  depend on this choice of representative, but  $\phi(g, h) = \phi_h(g)$  for  $m_G \otimes m_H$ -almost all pairs  $(g, h)$ . For every  $h \in H$  the function  $\phi_h$  is continuous (by continuity of  $\tilde{\psi}_0$  in the  $G$ -variable), bi- $K$ -invariant (by  $K$ -invariance of  $\tilde{\psi}_0$ ) and satisfies  $f * \phi_h = \hat{f}(\omega) \cdot \phi_h$  for every  $f \in \mathcal{H}(G, K)$  (since  $\tilde{\psi}$  and thus also  $\tilde{\psi}_0$  satisfy this property). It then follows from Lemma B.1, that  $\phi_h$  is a complex multiple of  $\omega$ . Consequently, there exists a measurable function  $c_\psi : H \rightarrow \mathbb{C}$  such that

$$\phi(g, h) = c_\psi(h) \cdot \omega(g).$$

Note that  $c_\psi$  is essentially bounded, since  $\tilde{\psi}$  and hence  $\phi(g, h)$  is. Now let  $f \in \mathcal{H}(G, K)$ , so that in particular  $f$  is right- $K$ -invariant. Using that  $\omega^* = \omega$  (see Lemma B.1) and that  $\text{covol}(\Gamma) = 1$  we compute

$$\begin{aligned}
\langle \mathcal{P}_\Gamma(f \otimes r), \psi \rangle &= \int_{G \times H} \overline{f(g) \cdot r(h)} \cdot \tilde{\psi}(g, h) dm_G(g) dm_H(h) \\
&= \int_{G \times H} \left( \int_K \overline{f(gk^{-1})} dm_K(k) \right) \overline{r(h)} \cdot \tilde{\psi}(g, h) dm_G(g) dm_H(h) \\
&= \int_{G \times H} \overline{f(g)} \cdot \overline{r(h)} \cdot \left( \int_K \tilde{\psi}(gk, h) dm_K(k) \right) dm_G(g) dm_H(h) \\
&= \int_{G \times H} \overline{f(g)} \cdot \overline{r(h)} \cdot \phi(g, h) dm_G(g) dm_H(h) \\
&= \int_{G \times H} \overline{f(g)} \cdot \overline{r(h)} \cdot c_\psi(h) \cdot \omega(g) dm_G(g) dm_H(h) \\
&= \int_G \overline{f(g) \omega^*(g^{-1})} dm_G(g) \cdot \int_H \overline{r(h)} \cdot c_\psi(h) dm_H(h) \\
&= \overline{\widehat{f}(\omega)} \cdot \langle r, c_\psi \rangle_H.
\end{aligned}$$

By Riesz' representation theorem it remains to show only that the linear functional

$$\lambda_r : L^2(Y)_\omega^K \rightarrow \mathbb{C}, \quad \psi \mapsto \langle r, c_\psi \rangle_H$$

is continuous. This follows from the fact that the linear map

$$\psi \mapsto c_\psi(h) \cdot \omega(g) = \phi(g, h) = \int_K \tilde{\psi}(gk, h) dm_K(k)$$

is bounded for almost every pair  $(g, h)$ . □

We can extend the shadow transform to compactly supported Riemann integrable functions by the usual approximation argument. This will be used in the following subsection.

**4.3. Pure-point diffraction for model sets associated with cocompact lattices.** We keep the setting of the previous subsection. In particular,  $\Gamma$  is a cocompact lattice in  $G \times H$ . Given a  $\Gamma$ -regular window  $W_0 \subset H$  we can express the spherical diffraction  $\hat{\eta}$  of the model set  $P_0 = P_0(G, H, \Gamma, W_0)$  in terms of the spectrum  $\Omega_Y$  of  $L^2(Y)_\omega^K$  and the shadow transform  $\widehat{\chi_{W_0}}$  of the characteristic function of the window. In the abelian case, this reduces to [1, Thm. 9.4], which in its essence goes back to the pioneering work of Meyer [38].

**Theorem 4.10** (Spherical diffraction formula). *Assume that  $\Gamma < G \times H$  is cocompact. Then, with notation as in the previous subsection, the spherical diffraction  $\hat{\eta}$  of the model set  $P_0 = P_0(G, H, \Gamma, W_0)$  is given by*

$$\hat{\eta} = \sum_{\omega \in \Omega_Y} \|\widehat{\chi_{W_0}}(\omega)\|_\omega^2 \cdot \delta_\omega.$$

*In particular,  $P_0$  is pure-point diffractive.*

*Proof of Theorem 4.10.* Let  $\{\psi_j \mid j \in J_\omega\}$  be an orthonormal basis for  $L^2(Y)_\omega^K$  so that

$$L^2(Y)^K = \bigoplus_{\omega \in \Omega_Y} L^2(Y)_\omega^K = \bigoplus_{\omega \in \Omega_Y} \bigoplus_{j \in J_\omega} \mathbb{C} \cdot \psi_j.$$



By Theorem 4.1 and Proposition 4.6 we then have for every  $f \in \mathcal{H}(G, K)$ ,

$$\begin{aligned}
\widehat{\eta}(|\widehat{f}|^2) &= \eta(f^* * f) = \|\mathcal{P}_\Gamma(f \otimes \chi_{W_0})\|_{L^2(Y)}^2 \\
&= \sum_{\omega \in \Omega_Y} \sum_{j \in J_\omega} |\langle \mathcal{P}_\Gamma(f \otimes \chi_{W_0}), \psi_j \rangle|^2 \\
&= \sum_{\omega \in \Omega_Y} \sum_{j \in J_\omega} \left| \widehat{f}(\omega) \cdot \langle \widehat{\chi_{W_0}}(\omega), \psi_j \rangle \right|^2 \\
&= \sum_{\omega \in \Omega_Y} \left( \sum_{j \in J_\omega} |\langle \widehat{\chi_{W_0}}(\omega), \psi_j \rangle|^2 \right) |\widehat{f}(\omega)|^2 \\
&= \sum_{\omega \in \Omega_Y} \|\widehat{\chi_{W_0}}(\omega)\|_\omega^2 \cdot \delta_\omega(|\widehat{f}|^2).
\end{aligned}$$

□

**4.4. The shadow transform as a generalized Hecke correspondence.** In order to make the spherical diffraction formula explicit, one needs to understand the spectrum  $\Omega_Y$  and the shadow transform; the former is a classical object of study, whereas the latter does not seem to appear in the present form in the literature yet. However, various special cases of the shadow transform have been studied before. Most notably, in the case where subgroup  $K < G$  is open, the shadow transform is closely related to the classical *Hecke correspondence* as we explain in this subsection.

To put things into perspective, let  $G$  be a totally-disconnected lcsc group,  $K < G$  a compact open subgroup so that  $(G, K)$  is a Gelfand pair,  $H$  another lcsc group and  $\Gamma < G \times H$  a lattice which projects densely onto both  $G$  and  $H$  and injectively to  $G$ . We will normalize the Haar measures on  $G$  and  $H$  such that  $m_G(K) = \text{covol}(\Gamma) = 1$ . Since  $\Gamma_G$  is dense in  $G$  and  $K$  is open, the multiplication map  $K \times \Gamma_G \rightarrow G$  is onto. We denote by  $g \mapsto (k_g, \gamma_g)$  a fixed Borel section of this map. As before we denote by  $\tau : \Gamma_G \rightarrow H$  the map  $\tau(g) = p_H(p_G^{-1}(g))$ .

**Proposition 4.11** (Hecke correspondence). *Let  $\Gamma_K := \Gamma \cap K$  and  $\Gamma_0 := \tau(\Gamma_K) < H$ .*

- (i)  $\Gamma_0 < H$  is a lattice, which is cocompact if and only if  $\Gamma$  is cocompact.
- (ii) The map  $j : K \backslash (G \times H) / \Gamma \rightarrow H / \Gamma_0$  given by  $j([g, h]) := h\tau(\gamma_g^{-1})\Gamma_0$  is a homeomorphism with inverse given by  $i : H / \Gamma_0 \rightarrow K \backslash (G \times H) / \Gamma$ ,  $h\Gamma_0 \mapsto K(e, h)\Gamma$ .
- (iii)  $i$  and  $j$  induce mutually inverse isomorphisms of  $H$ -representations

$$i^* : L^2(Y)^K \rightarrow L^2(H / \Gamma_0) \quad \text{and} \quad j^* : L^2(H / \Gamma_0) \rightarrow L^2(Y)^K.$$

- (iv) The Hecke algebra  $\mathcal{H}(G, K)$  acts on  $L^2(H / \Gamma_0)$  via

$$T(\rho)(f)(h\Gamma_0) = \int_G \rho(g) f(\tau(\gamma_g)h\Gamma_0) dm_G(g) \quad (\rho \in \mathcal{H}(G, K), f \in L^2(H / \Gamma_0)).$$

*Proof.* We first prove (ii). Observe first that for all  $(g, h) \in G \times H$ ,

$$K(g, h)\Gamma = K(k_g\gamma_g, h)\Gamma = K(\gamma_g, h)\Gamma = K(e, h\tau(\gamma_g)^{-1})\Gamma.$$

This shows that the map  $\pi : H \rightarrow K \backslash G \times H / \Gamma$  given by  $\pi(h) := K(e, h)\Gamma$  is onto. Now assume that  $\pi(h_1) = \pi(h_2)$ . Then

$$\exists k \in K, \gamma \in \Gamma_G : (e, h_1) = k(e, h_2)(\gamma, \tau(\gamma)) = (k\gamma, h_2\tau(\gamma)) \quad (4.3)$$

This implies that  $k\gamma = e$ , hence  $k = \gamma^{-1} \in \Gamma_K$  and thus  $h_1 \in h_2\Gamma_0$ . Conversely, if  $h_1 \in h_2\Gamma_0$  then (4.3) holds. Thus  $\pi$  factors through  $H / \Gamma_0$  and defines a continuous bijection  $i$  as in the proposition with inverse  $j$ . Now note that  $H$  acts on  $K \backslash (G \times H) / \Gamma$ , since it commutes with  $K$ , and that  $i$  is  $H$ -equivariant. It follows that  $i$  is open, whence  $i$  and  $j$  are mutually inverse homeomorphisms. This proves (ii) and shows in particular that  $\Gamma_0$  is of finite covolume, respectively

cocompact in  $H$  if and only if  $\Gamma < G \times H$  has the corresponding property. To show (i) it thus remains to show only that  $\Gamma_0$  is discrete. However, for every compact subset  $W \subset H$  we have

$$\Gamma_0 \cap W = \tau(\Gamma_K) \cap W = \tau(p_G((K \times W) \cap \Gamma)),$$

which is finite by discreteness of  $\Gamma$ . This finishes the proof of (i) and provides us with a unique  $H$ -invariant probability measure  $m_{H/\Gamma_0}$  on  $H/\Gamma_0$ . Now (ii) yields an  $H$ -equivariant isomorphism  $i^* : C_c(Y)^K \rightarrow C_c(H/\Gamma_0)$ , and under this identification the  $H$ -invariant measure  $m_Y$  on  $Y$  must correspond to  $m_{H/\Gamma_0}$ , hence we deduce that (iii) holds. In particular,  $\mathcal{H}(G, K)$  acts on  $L^2(H/\Gamma_0)$  via

$$T(\rho)(f) := i^*(\pi_Y(\rho).(j^*f)) \quad (\rho \in \mathcal{H}(G, K), f \in L^2(H/\Gamma_0)).$$

Writing out the definitions of  $i^*$ ,  $j^*$  and  $\pi_Y$  explicitly we end up with (iv).  $\square$

**Example.** The most classical examples of a Hecke correspondence is given by  $G := \mathrm{PGL}_2(\mathbb{Q}_p)$ ,  $K := \mathrm{PGL}_2(\mathbb{Z}_p)$ ,  $H := \mathrm{PGL}_2(\mathbb{R})$  and  $\Gamma := \mathrm{PGL}_2(\mathbb{Z}[1/p])$  sitting diagonally inside  $G \times H$ . In this case,  $\Gamma_K = \mathrm{PGL}_2(\mathbb{Z})$  and thus

$$\Lambda_2 := H/\Gamma_0 = \mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$$

can be identified with the space of homothety classes of lattices in  $\mathbb{R}^2$ . Here the Hecke algebra  $\mathcal{H}(G, K)$  is generated by the single element

$$\rho_0 := \chi_{Kg_pK} \in \mathcal{H}(G, K), \quad \text{where } g_p := \begin{pmatrix} p & \\ & 1 \end{pmatrix},$$

and the action of the Hecke algebra on  $\Lambda_2$  is determined by

$$T(\rho_0)f(\Lambda) = \sum_{[\Lambda:\Lambda']=p} f(\Lambda').$$

Let us now return to the case where  $\Gamma$  is cocompact. Then  $\Gamma_0$  is cocompact, and we can decompose the representation  $(L^2(H/\Gamma_0), T)$  of  $\mathcal{H}(G, K)$  as

$$L^2(H/\Gamma_0) = \bigoplus_{\omega \in \Omega_Y} L^2(H/\Gamma_0)_\omega,$$

where  $L^2(H/\Gamma_0)_\omega \subset L^2(H/\Gamma_0)$  denotes the  $\mathcal{H}(G, K)$ -eigenspace of weight  $\omega$ . Given a function  $r \in C_c(H)$ , we denote by  $\mathcal{P}_{\Gamma_0}r \in C_c(H/\Gamma_0) \subset L^2(H/\Gamma_0)$  its periodization over  $\Gamma_0$  and by  $\mathcal{P}_{\Gamma_0}r_\omega$  the projection of the latter onto  $L^2(H/\Gamma_0)_\omega$ .

**Proposition 4.12** (Shadow transform vs. Hecke correspondence). *Let  $r \in C_c(H/\Gamma_0)$  and denote by  $j^* : L^2(H/\Gamma_0) \rightarrow L^2(Y)^K$  the Hecke correspondence. Then*

$$\widehat{r}(\omega) = j^*(\mathcal{P}_{\Gamma_0}r_\omega).$$

*Proof.* Since  $\chi_K$  is a two-sided identity in  $\mathcal{H}(G, K)$  we have for all  $f \in \mathcal{H}(G, K)$  and  $\omega \in \mathcal{S}^+(G, K)$ ,

$$\widehat{\chi_K}(\omega) \cdot \widehat{f}(\omega) = \widehat{\chi_K * f}(\omega) = \widehat{f}(\omega),$$

and hence  $\widehat{\chi_K}(\omega) = 1$ . We thus deduce from Corollary 4.8 that  $\widehat{r}(\omega) = \pi_\omega(\mathcal{P}_\Gamma(\chi_K \otimes r))$ . Now for  $(g, h) \in G \times H$  we have

$$\begin{aligned} \mathcal{P}_\Gamma(\chi_K \otimes r)((g, h)\Gamma) &= \sum_{\gamma \in \Gamma_G} \chi_K(g\gamma)r(h\tau(\gamma)) = \sum_{\gamma \in \Gamma_G} \chi_K(k_g\gamma_g\gamma)r(h\tau(\gamma)) \\ &= \sum_{\gamma \in \Gamma_G} \chi_K(\gamma_g\gamma)r(h\tau(\gamma)) = \sum_{\gamma \in \gamma_g\Gamma_G} \chi_K(\gamma)r(h\tau(\gamma_g^{-1})\tau(\gamma)) \\ &= \sum_{\gamma \in \Gamma_G} \chi_K(\gamma)r(j([g, h])\tau(\gamma)) = \sum_{\gamma \in \Gamma_K} r(j([g, h])\tau(\gamma)) \\ &= \mathcal{P}_{\Gamma_0}(r \circ j)((g, h)\Gamma) = j^*(\mathcal{P}_{\Gamma_0}r)((g, h)\Gamma), \end{aligned}$$

hence  $\widehat{r}(\omega) = \pi_\omega(j^*(\mathcal{P}_{\Gamma_0} r))$ . Finally, by  $\mathcal{H}(G, K)$ -equivariance of the Hecke-correspondence, we have

$$\widehat{r}(\omega) = \pi_\omega(j^*(\mathcal{P}_{\Gamma_0} r)) = j^*(\pi_\omega(\mathcal{P}_{\Gamma_0} r)) = j^*(\mathcal{P}_{\Gamma_0} r_\omega).$$

□

**Corollary 4.13** (Spherical diffraction formula for a compact-open  $K$ ). *In the situation of Proposition 4.12 the spherical diffraction  $\widehat{\eta}$  of the model set  $P_0 = P_0(G, H, \Gamma, W_0)$  is given by*

$$\widehat{\eta} = \sum_{\omega \in \Omega_Y} \|\pi_\omega(\mathcal{P}_{\Gamma_0} \chi_{W_0})\|_{H/\Gamma_0}^2 \cdot \delta_\omega.$$

□

## 5. APPROXIMATIONS OF THE SPHERICAL AUTO-CORRELATION

**5.1. A criterion for genericity of the Hecke algebra.** We now return to the general case, where  $P_0 = P_0(G, H, \Gamma, W_0)$  is a model set associated with an arbitrary (i.e. not necessarily co-compact) lattice  $\Gamma < G \times H$ . We assume that we are given a compact subgroup  $K < G$  such that  $(G, K)$  is a Gelfand pair. The general goal of this section is to make the abstract approximation theorem for the auto-correlation (Theorem 2.10) more explicit in this setting. We are particularly interested in the *spherical auto-correlation*  $\eta_{\text{sph}}$ , i.e. the restriction of the auto-correlation  $\eta$  of  $P_0$  to the Hecke algebra  $\mathcal{H}(G, K)$ , since this determines the spherical diffraction of  $P_0$ .

According to Theorem 2.10 we have

$$\eta_{\text{sph}}(f) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0} f(x^{-1}y) \quad (5.1)$$

for all  $f \in \mathcal{H}(G, K)$  provided  $(F_t)$  is a weakly admissible sequence of compact subsets of  $G$  and  $\mathcal{H}(G, K)$  is generic (in the sense of Definition 2.9) with respect to  $(F_t)$  and the invariant measure  $\nu$  on  $X$ . We are going to make the latter condition more explicit.

As before, we denote  $Y := (G \times H)/\Gamma$ . Given a sequence  $(F_t)$  of compact subsets  $G$  of positive measure we denote by  $(\beta_t)$  the associated sequence of probability measures on  $G$  defined by

$$\beta_t(\rho) = \frac{1}{m_G(F_t)} \int_{F_t} \rho(s) dm_G(s), \quad \text{for } \rho \in C_c(G).$$

Note that the definition of genericity involves the  $G$ -action on  $X$ . We will replace it by a condition which only involves the  $G$ -action on  $Y$ .

**Definition 5.1.** A sequence  $(\beta_t)$  of bi- $K$ -invariant Borel probability measures on  $G$  is called *pointwise good with respect to  $(K, Y)$*  if

$$\lim_{t \rightarrow \infty} \int_G h(g^{-1} \cdot y) d\beta_t(g) = \int_Y h dm_Y, \quad \text{for all } h \in C_b(Y)^K \text{ and } y \in Y.$$

**Lemma 5.2** (Genericity criterion). *Let  $K < G$  be a compact subgroup and suppose that  $(F_t)$  is a sequence of compact subsets  $G$  of positive measure. If the associated sequence  $(\beta_t)$  of probability measures on  $G$  is pointwise good with respect to  $(Y, K)$ , then  $\mathcal{H}(G, K)$  is generic with respect to  $\nu$  and  $(F_t)$ .*

*Proof.* Given  $f_1, f_2 \in \mathcal{H}(G, K)$  let us denote by  $h : Y \rightarrow \mathbb{C}$  the function

$$h(y) := \overline{\mathcal{P}_\Gamma(f_1 \otimes \chi_{W_0})(y)} \mathcal{P}_\Gamma(f_2 \otimes \chi_{W_0})(y),$$

and let  $y_0$  denote the image of  $P_0$  under the parametrization map  $\beta : X \rightarrow Y$ . By Lemma 2.2 and the pointwise good assumption we have

$$\begin{aligned}
\eta(f_1^* * f_2) &= \langle \mathcal{S}f_1, \mathcal{S}f_2 \rangle_X = \langle \mathcal{P}_\Gamma(f_1 \otimes \chi_{W_0}), \mathcal{P}_\Gamma(f_2 \otimes \chi_{W_0}) \rangle_Y \\
&= \int_Y h dm_Y = \lim_{t \rightarrow \infty} \int_G h(g^{-1} \cdot y_0) d\beta_t(g) \\
&= \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \int_{F_t} h(s^{-1} \cdot y_0) dm_G(s) \\
&= \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \int_{F_t} \overline{\mathcal{P}_\Gamma(f_1 \otimes \chi_{W_0})(s^{-1} \cdot y_0)} \mathcal{P}_\Gamma(f_2 \otimes \chi_{W_0})(s^{-1} \cdot y_0) dm_G(s) \\
&= \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \int_{F_t} \overline{\mathcal{S}f_1(s^{-1} \cdot P_0)} \mathcal{S}f_2(s^{-1} \cdot P_0) dm_G(s).
\end{aligned}$$

□

**5.2. Reduction to weak ergodicity.** The genericity criterion of the last subsection can be reduced to a weak pointwise ergodic theorem on  $Y = (G \times H)/\Gamma$ .

**Definition 5.3.** A sequence  $(\beta_t)$  of bi- $K$ -invariant Borel probability measures on  $G$  is called *weakly ergodic* with respect to  $(K, Y)$  if for every  $f \in L^2(Y)^K$  the functions

$$F_t(y) := \int_G f(s^{-1}y) d\beta_t(s)$$

converge to the constant function  $\int_Y f dm_Y$  with respect to the weak topology in  $L^2(Y)$  as  $t \rightarrow \infty$ .

We recall from Proposition B.8 that weak ergodicity admits the following spectral representation. We can write the  $\mathcal{H}(G, K)$ -representation  $L^2(Y)^K$  as a direct integral

$$L^2(Y)^K = \int_{\text{spec}_{(G, K)}(L^2(Y))} V_\omega d\nu_Y(\omega),$$

where  $V_\omega$  denotes the irreducible spherical representation with associated spherical function  $\omega$ . Then the *spherical spectrum* of  $Y$  is  $\text{spec}_{(G, K)}(Y) := \text{supp}(\nu_Y)$ . Then  $(\beta_t)$  is weakly ergodic if and only if

$$\widehat{\beta_t}(\omega) \rightarrow 0 \quad \text{for all } \omega \in \text{spec}_{(G, K)}(Y) \setminus \{1\}.$$

The following theorem says that the spherical mean ergodic theorem (Proposition B.8) implies the corresponding pointwise statement. The work of Gorodnik and Nevo [18, 19, 20, 21, 22] investigates in great generality under which conditions one can sharpen mean ergodic theorems for non-amenable groups into pointwise statements. Our proof is based on an adaptation of their ideas to the setting at hand.

**Theorem 5.4** (Weak ergodicity implies goodness). *If a sequence  $(\beta_t)$  of bi- $K$ -invariant Borel probability measures on  $G$  is weakly ergodic with respect to  $(K, Y)$ , then it is pointwise good with respect to  $(K, Y)$ .*

The proof of Theorem 5.4 is based on two lemmas. The first one is just a convenient reformulation of Definition 5.3.

**Lemma 5.5.** *Suppose that  $\nu$  is a bounded positive Borel measure on  $Y$  which is absolutely continuous with respect to  $m_Y$  with a square-integrable density. If  $(\beta_t)$  is weakly ergodic with respect to  $(K, Y)$ , then  $\check{\beta}_t * \nu \rightarrow m_Y$  in the vague topology.*

The second ingredient is the following continuity result, which is known in various related forms and which we state and prove here for ease of reference.

**Lemma 5.6.** *Let  $\psi \in C_c(G \times H)$  be non-negative and fix  $y \in Y$ . Then the bounded non-negative measure  $\nu$  on  $Y$ , defined by*

$$\nu(f) = (\psi * \delta_y)(f) = \int_G \psi(g)f(gy)dm_G(g) \quad \text{for } f \in C_c(Y),$$

*is absolutely continuous with respect to  $m_Y$  with a bounded density.*

*Proof.* It suffices to prove the lemma when  $y = (e, e)\Gamma$ . We begin by noting that  $\nu$  is uniquely determined by its values at functions  $f$  of the form

$$f((g, h)\Gamma) = \sum_{\gamma \in \Gamma} F((g, h)\gamma), \quad \text{where } F \in C_c(G \times H),$$

and a straightforward calculation shows that if  $f$  has this form, then

$$\nu(f) = \int_G F(g, h)\rho_\psi((g, h)\Gamma)dm_G(g)dm_G(h) = \int_Y f(y)\rho_\psi(y)dm_Y(y),$$

where

$$\rho_\psi((g, h)\Gamma) = \sum_{\gamma \in \Gamma} \psi((g, h)\gamma).$$

This shows that  $\nu$  is absolutely continuous with respect to  $m_Y$  with density  $\rho_\psi$ . To prove that this density is bounded, we observe that

$$|\rho_\psi((g, h)\Gamma)| \leq \|\psi\|_\infty \cdot |\text{supp}(\psi) \cap (g, h)\Gamma|$$

Since  $\Gamma$  has finite local complexity and  $\text{supp}(\psi)$  is compact, the right-hand side is uniformly bounded (see Lemma A.13).  $\square$

*Proof of Theorem 5.4.* Given a non-negative  $\psi \in C_c(G \times H)$  and a bounded positive measure  $\nu$  on  $Y$ , we define a new bounded positive measure  $\psi * \nu$  by

$$\psi * \nu(\phi) := \nu(\check{\psi} * \phi).$$

With this notation we have to show that for every  $y \in Y$  the sequence  $(\check{\beta}_t)_* \delta_y$  converges to  $m_Y$  in the vague topology. Note that the set  $\{(\check{\beta}_t)_* \delta_y\}$  is vaguely sequentially pre-compact in the space of all sub-probability measures. It thus suffices to show that every vague limit point  $\nu$  coincides with  $m_Y$ .

Thus assume that  $\nu$  is a sub-probability measure given by  $\nu = \lim(\check{\beta}_{t_n})_* \delta_y$  for some sub-sequence  $(t_n)$  and note that  $\nu$  is automatically  $K$ -invariant. We have to show that  $\nu = m_Y$ . For this it suffices to show that  $\rho * \nu = m_Y$  for every probability density  $\rho \in C_c(G \times H)^K$ , for then we can extract an approximate identity  $(\rho_n)$  of such functions and conclude that

$$\nu = \lim_{n \rightarrow \infty} (\rho_n)_* \nu = m_Y.$$

Thus let  $\rho \in C_c(G \times H)^K$  be a probability density and  $f \in C_c(Y)$ . Observe that  $\check{\beta}_t$  commutes with  $C_c(H)$  since the  $G$  and  $H$  actions commute, and thus, by commutativity of the Hecke algebra, the  $\check{\beta}_t$  commute with every  $\rho \in C_c(G \times H)^K$  with respect to convolution. We thus have

$$\begin{aligned} \rho * \nu(f) &= \lim_{n \rightarrow \infty} \rho_*((\check{\beta}_{t_n})_* \delta_y)(f) = \lim_{n \rightarrow \infty} (\rho * \check{\beta}_{t_n})_* \delta_y(f) \\ &= \lim_{n \rightarrow \infty} (\check{\beta}_{t_n} * \rho)_* \delta_y(f) = \lim_{n \rightarrow \infty} (\rho * \delta_y)(\beta_{t_n} * f). \end{aligned}$$

By Lemma 5.6,  $\rho * \delta_y$  is absolutely continuous with respect to  $m_Y$  with a bounded (hence square-integrable) density. By Lemma 5.5, we conclude that the last limit above is equal to  $m_Y(f)$ , which finishes the proof.  $\square$

**Remark 5.7.** In Theorem 5.4 we deduced a pointwise ergodic theorem for  $K$ -invariant functions with respect to the weak\*-topology on  $C_b(Y)^*$  from the corresponding mean ergodic theorem. If one instead assumes that  $(\beta_t)$  is weakly admissible and satisfies the pointwise ergodic theorem in  $L^2(Y)$  (in the sense of [18]), then one can deduce from the *invariance principle* ([18, Theorem 5.22]) that  $(\beta_t * f)(y) \rightarrow m_Y(f)$  for every  $y \in Y$  and all  $f \in C_b(Y)$  (without assuming  $K$ -invariance). This is a stronger statement than Theorem 5.4, but under a slightly stronger hypothesis.

Let us illustrate how to deduce convergence in  $C(Y)^*$  in the compact case, assuming the pointwise ergodic theorem in  $L^2(Y)$  (which holds in many examples of interest, see [18]). The invariance principle says that for every  $f \in C(Y)$ , there exists a  $G$ -invariant and  $m_Y$ -conull subset  $Y_f \subset Y$  such that  $(\beta_t * f)(y) \rightarrow m_Y(f)$  for all  $y \in Y_f$ . By  $\sigma$ -compactness and second countability of  $Y$ , we can find a countable dense subset  $\mathcal{C} \subset C(Y)$ ; and thus the set  $Y_o = \bigcap_{f \in \mathcal{C}} Y_f$  is both  $G$ -invariant and  $m_Y$ -conull. By a simple approximation argument, we see that,  $(\beta_t * f)(y) \rightarrow m_Y(f)$  for all  $y \in Y_o$  and for every  $f \in C(Y)$ . In particular, for any  $f \in C(Y)$ , we have

$$(\beta_t * f)(h.y) = (\beta_t * f_h)(y) \rightarrow m_Y(f_h) = m_Y(f),$$

where  $f_h(y) = f(h.y)$  and  $h \in H$ . This shows that  $Y_o$  is not only  $G$ -invariant, but also  $H$ -invariant. However, the  $G \times H$ -action on  $Y$  is transitive and thus  $Y_o = Y$ , which shows that we have  $\check{\beta}_t * \delta_y \rightarrow m_Y$  for every  $y \in Y$  in the weak\*-topology on  $C(Y)^*$ .

Motivated by Theorem 2.10 we define:

**Definition 5.8.** Let  $(F_t)$  be a sequence of bi- $K$ -invariant compact subsets of  $G$  of positive Haar measure. Then  $(F_t)$  is called a *good approximation sequence* for  $\Gamma$  with respect to the Gelfand pair  $(G, K)$  if

- (i)  $(F_t)$  is weakly admissible;
- (ii) the associated sequence  $(\beta_t)$  of bi- $K$ -invariant Borel probability measures on  $G$  satisfies

$$\lim_{t \rightarrow \infty} \widehat{\beta}_t(\omega) = 0 \quad \text{for all } \omega \in \text{spec}_{(G, K)}((G \times H)/\Gamma) \setminus \{1\}.$$

Combining Theorem 2.10, Lemma 5.2 and Theorem 5.4 with the spectral characterization of weak ergodicity (Proposition B.8) we obtain:

**Theorem 5.9** (Approximation theorem for model sets). *Let  $(F_t)$  be a good approximation sequence for  $\Gamma$  with respect to the Gelfand pair  $(G, K)$  and let  $P_0 = P_0(G, H, \Gamma, W_0)$  be a model set with spherical auto-correlation  $\eta_{\text{sph}}$ . Then for all  $f \in \mathcal{H}(G, K)$ ,*

$$\eta_{\text{sph}}(f) = \lim_{t \rightarrow \infty} \frac{1}{m_G(F_t)} \sum_{x \in P_0 \cap F_t} \sum_{y \in P_0} f(x^{-1}y).$$

□

## 6. EXAMPLES

We conclude this article by discussing various classes of examples of model set triples  $(G, H, \Gamma)$ . We then discuss in each case possible Gelfand pairs  $(G, K)$  and corresponding good approximation sequences for  $\Gamma$ . We also point out in each case specific properties which distinguish the corresponding model sets from model sets in abelian groups. Given an model set triple  $(G, H, \Gamma)$  we will abbreviate  $Y := (G \times H)/\Gamma$ .

**6.1. Model sets in amenable groups.** There are plenty of examples of model set triples  $(G, H, \Gamma)$  with  $G$  amenable, but requiring that  $G$  is part of Gelfand pair  $(G, K)$  with  $K$  compact is a rather restrictive condition. For example, in the case of connected Lie groups we have the following:

**Proposition 6.1** (Model set triples of amenable Lie groups). *Let  $(G, H, \Gamma)$  be a model set triple, where  $G$  and  $H$  are Lie groups and  $G$  is amenable and part of a Gelfand pair  $(G, K)$  with  $K$  compact,  $G/K$  simply-connected and  $G$  acting effectively on  $G/K$ . Then the following hold:*

- (i)  $G = N \rtimes L$ , where  $L$  is compact and  $N$  is either abelian or 2-step nilpotent.
- (ii) If  $\pi_G(\Gamma) < N$ , then  $H$  is 2-step nilpotent and  $\Gamma$  is cocompact.

*Proof.* (i) is an immediate consequence of Vinberg's decomposition theorem [55, Thm. 13.3.20]. If  $\pi_G(\Gamma) < N$ , then  $\pi_G(\Gamma)$  and thus  $\Gamma$  are 2-step nilpotent. But then also  $H$  is 2-step nilpotent since it contains the dense 2-step nilpotent subgroup  $\pi_H(\Gamma)$ , and thus  $\Gamma$  is cocompact by [46, Thm. 2.1].  $\square$

In the situation of Proposition 6.1 the possible pairs  $(G, L)$  appearing in (i) can actually be classified; these are called *nilmanifold pairs* (see [55, Sec. 13.4]). If we assume additionally that  $\pi_G(\Gamma) < N$ , then we obtain a full classification of all possible triples  $(G, H, \Gamma)$ . Namely,  $H$  has to be a 2-step nilpotent Lie group, and these are well-known. Then  $\Gamma$  has to be a lattice in the nilpotent Lie group  $N \times H$ , and hence arises from a rational basis of the Lie algebra of  $N \times H$  by the construction described in [46, Remark after Thm. 2.12]. Since the details of the classification are long and technical, we confine ourselves to a generic example.

**Example.** Let  $N := H_n(\mathbb{R}) = \mathbb{C}^n \ltimes \mathbb{R}$ , the  $(2n+1)$ -dimensional real Heisenberg group,  $K < L$  subgroups of  $U(n)$  containing a maximal torus (acting on  $N$  by rotations on the  $\mathbb{C}^n$  part),  $G := N \rtimes L$  and  $H := H_n(\mathbb{R})$ . There are uncountably many lattices in  $N \times H = H_n(\mathbb{R}) \times H_n(\mathbb{R})$  (parametrized by rational bases of the Lie algebra of  $H_n(\mathbb{R}) \times H_n(\mathbb{R})$ ), and by Proposition 6.1 all of them are cocompact. It is easy to check for a concrete lattice  $\Gamma$  whether  $(G, H, \Gamma)$  is a model set triple, and indeed this is the case generically.

If we drop the condition that  $\pi_G(\Gamma) < N$  then we can no longer classify the corresponding model set triples. For example, the following problem seems to be open even in the case where  $G$  and  $H$  are Lie groups.

**Problem.** If  $(G, H, \Gamma)$  is a model set triple with  $H$  connected,  $G$  connected and amenable and part of a Gelfand pair  $(G, K)$  with  $K$  compact, is  $\Gamma$  cocompact in  $G \times H$ ?

If we also drop the connectedness assumptions on  $G$  and  $H$ , then the answer to this question becomes negative. In fact, there exist totally disconnected compact-by-abelian groups  $G$  and  $H$  such that  $G$  is part of a Gelfand pair  $(G, K)$  with compact  $K$  and such that there is a model set triple  $(G, H, \Gamma)$  with  $\Gamma < G \times H$  non-cocompact. The following example is based on the construction of a non-cocompact lattice in a compact-by-abelian group due to Bader, Caprace, Gelfand and Mozes [7, Example 3.5].

**Example.** Let  $S$  be a set of primes which is “thin” in the sense that  $\sum_{p \in S} \frac{1}{p} < \infty$  and set

$$V := \bigoplus_{p \in S} \mathbb{F}_p \quad \text{and} \quad K := \prod_{p \in S} \mathbb{F}_p^\times,$$

so that  $V$  is abelian and  $K$  is compact and acts by coordinate-wise multiplication on  $V$ . Let  $G := H := V \rtimes K$  so that  $G$  and  $H$  are totally disconnected and compact-by-abelian and  $(G, K)$  is a Gelfand pair. Given  $\gamma \in G$  we write  $\gamma = (\gamma_p)_{p \in S}$ , where  $\gamma_p \in \mathbb{F}_p \rtimes \mathbb{F}_p^\times$ . We define a dense subgroup  $\Gamma_0 < G$  by

$$\Gamma_0 := \bigoplus_{p \in S} \mathbb{F}_p \rtimes \mathbb{F}_p^\times < G,$$

and a dense embedding  $\tau : \Gamma_0 \rightarrow H$  by

$$\tau((b_p, a_p)_{p \in S}) = ((1, 1)(b_p, a_p)(1, 1)^{-1})_{p \in S} = (b_p + 1 - a_p, a_p)_{p \in S}.$$

**Proposition 6.2.** *The subgroup  $\Gamma := \{(\gamma, \tau(\gamma)) \mid \gamma \in \Gamma_0\} < G \times H$  is a non-cocompact lattice (and  $(G, H, \Gamma)$  is a model set triple).*

*Proof.* To see that  $\Gamma$  is discrete, suppose that  $(\gamma^j)$  is a sequence in  $\Gamma_0$  such that  $(\gamma^j, \tau(\gamma^j)) \rightarrow e$  in  $\Gamma$ , and let  $\gamma_p^j = (a_p^j, b_p^j)$  so that

$$b_p^j \rightarrow 0 \quad \text{and} \quad b_p^j + 1 - a_p^j \rightarrow 0 \quad \text{for all } p \in S.$$

Since  $\mathbb{F}_p$  is finite, the former implies that  $b_p^j = 0$  for all sufficiently large  $j$  (depending on  $p$ ), and then the latter implies that  $a_p^j = 1$  for all such  $j$ . Thus  $(\gamma^j, \tau(\gamma^j)) = (e, e)$  eventually, and thus  $\Gamma$  is discrete. To show that  $\Gamma$  has finite covolume it suffices to show that

$$\sum_{[(u,v)] \in \Gamma \backslash V \oplus V} \frac{1}{|\Gamma_{(u,v)}|} < \infty,$$

where  $\Gamma_{(u,v)}$  denotes the stabilizer of  $(u, v) \in V \oplus V$  in  $\Gamma$  and the sum is over all  $\Gamma$ -orbits in  $V \oplus V$  (see [6, Sec. 1.5]). Unravelling definitions we see that two points  $(u, v), (u', v') \in V \oplus V$  are in the same  $\Gamma$ -orbit if and only if there exist for every  $p \in S$  elements  $b_p \in \mathbb{F}_p$  and  $a_p \in \mathbb{F}_p^\times$  such that

$$b_p + a_p u_p = u'_p \quad \text{and} \quad b_p + 1 - a_p + a_p v_p = v'_p. \quad (6.1)$$

We claim that all  $\Gamma$ -orbits in  $V \oplus V$  are exactly the sets

$$S_I := \{(u, v) \mid v_p - u_p = 1 \Leftrightarrow p \in I\},$$

where  $I \subset S$  is a finite subset. Indeed, let  $(u, v) \in V \oplus V$  and set  $I := \{p \in S \mid v_p - u_p = 1\}$  so that  $(u, v) \in S_I$ . Assume first that  $(u', v') \in \Gamma \cdot (u, v)$  and let  $a_p, b_p$  as in (6.1). Then

$$v'_p - u'_p = b_p + 1 - a_p + a_p v_p - b_p - a_p u_p = 1 + a_p(v_p - u_p - 1),$$

hence  $v'_p - u'_p = 1$  if and only if  $v_p - u_p = 1$ , hence  $(u', v') \in S_I$ . Conversely, assume that  $(u', v') \in S_I$ . We then need to find  $(a_p, b_p)$  satisfying (6.1) for every  $p \in S$ . If  $p \in I$  we set  $a_p = 1$  and  $b_p := u'_p - u_p$ . Then (6.1) holds since  $v_p - u_p = v'_p - u'_p = 1$  and thus

$$b_p + a_p u_p = (u'_p - u_p) + 1 \cdot u_p = u'_p \quad \text{and} \quad b_p + 1 - a_p + a_p v_p = (u'_p - u_p) + 1 - 1 + v_p = u'_p + 1 = v'_p.$$

If  $p \notin I$ , then  $w_p := v_p - u_p - 1$  and  $w'_p := v'_p - u'_p - 1$  are non-zero, hence we may define  $a_p := w'_p w_p^{-1}$  and  $b_p := u'_p - a_p u_p$ . We then compute

$$b_p + a_p u_p = u'_p - a_p u_p + a_p u_p = u'_p$$

and

$$b_p + 1 - a_p + a_p v_p = u'_p - a_p u_p + 1 - a_p + a_p v_p = u'_p + 1 + a_p w_p = u'_p + 1 + w'_p = v'_p,$$

which establishes (6.1) also in this case and finishes the proof of the claim.

Given  $(u, v) \in S_I$  we have

$$\Gamma_{(u,v)} = \bigoplus_{p \in S} \{(b_p, a_p) \in \mathbb{F}_p \times \mathbb{F}_p^\times \mid b_p + a_p u_p = u_p \quad \text{and} \quad b_p + 1 - a_p + a_p v_p = v_p\}$$

Subtracting the two equations yields

$$v_p - u_p - 1 = a_p(v_p - u_p - 1),$$



which has a unique solution  $a_p$  if  $p \notin I$  and  $(p-1)$  solutions for  $a_p$  otherwise. Given  $a_p$ , there is a unique  $b_p$  satisfying the first equation, hence  $|\Gamma_{(u,v)}| = \prod_{p \in I} (p-1)$ . We obtain

$$\sum_{[(u,v)] \in \Gamma \backslash V \oplus V} \frac{1}{|\Gamma_{(u,v)}|} = \sum_{\substack{I \subset S \\ \text{finite}}} \frac{1}{\prod_{p \in I} (p-1)} < \infty,$$

which shows that  $\Gamma$  is a lattice in  $G \times H$ . On the other hand, since there are infinitely many  $\Gamma$ -orbits in  $V \oplus V$ , we deduce that  $\Gamma$  is not cocompact.  $\square$

This shows that amenable model set triples can be more complicated than the examples from Lie groups suggest. Still, as far as good approximation sequences for the auto-correlation are concerned, any weakly admissible Følner sequence will do, and hence the theory of auto-correlation for amenable model sets is very close to the abelian case.

**6.2. Model sets in semisimple  $S$ -adic groups.** Let  $\mathbb{G}$  be a semisimple algebraic group defined over  $\mathbb{Q}$  and assume that  $\mathbb{G}(\mathbb{R})$  splits non-trivially as a product of semisimple Lie groups over  $\mathbb{R}$ , say  $\mathbb{G}(\mathbb{R}) = G \times H_0$ . Then the arithmetic group  $\Gamma_0 = \mathbb{G}(\mathbb{Z})$  is a lattice in  $G \times H_0$ , and there are explicit conditions on  $\mathbb{G}$  which ensure that  $\Gamma_0$  is irreducible, i.e. projects densely into both factors (see [45]). Assuming irreducibility of  $\Gamma_0$ , we obtain that  $\Gamma_1 := \ker(p_G|_{\Gamma_0})$  is a normal subgroup of  $\Gamma_0$ . Note that  $\Gamma_1$  is of infinite index in  $\Gamma_0$  since  $\Gamma := \Gamma_0/\Gamma_1$  projects densely to  $G$ . It thus follows from Margulis' normal subgroup theorem (which applies since  $\text{rk}_{\mathbb{R}}(G \times H_0) \geq 2$ ) that  $\Gamma_1$  is finite. Thus if we define  $H := H_0/p_H(\Gamma_1)$ , then  $\Gamma$  is a lattice in  $G \times H$  and  $(G, H, \Gamma)$  is a model set triple. Using this construction we can construct model sets of both compact and non-compact type in any semisimple linear algebraic group  $G$  over  $\mathbb{R}$ . Non-arithmetic examples can be constructed similarly in  $\text{SO}(n, 1)$  and  $\text{SU}(n, 1)$ . The whole construction can also be extended to the  $S$ -adic setting (see e.g. [7] for basic definitions concerning the notion of  $S$ -adic groups) by starting from  $S$ -arithmetic groups. We deduce:

**Proposition 6.3** (Model sets in semisimple  $S$ -adic groups). *Every semisimple  $S$ -adic group admits model sets of both compact and non-compact type.*  $\square$

Given a semisimple  $S$ -adic group  $G$  we can define a maximal compact subgroup  $K$  of  $G$  by taking the product of special maximal compact subgroups of the local factors. For each Archimedean local factor we choose an arbitrary maximal compact subgroup, and for each non-Archimedean local factor we take a vertex stabilizer in the associated Bruhat-Tits building. The resulting product group  $K$  is independent of the choices up to conjugation, and  $(G, K)$  is a Gelfand pair. There are plenty of weakly admissible sequences  $(F_t)$  for the Gelfand pair  $(G, K)$  (see [18]), and hence there are plenty of good approximation sequences for model sets triples  $(G, H, \Gamma)$  by the following proposition.

**Proposition 6.4** (Good approximation sequences in semisimple  $S$ -adic groups). *Let  $G$  be an  $S$ -adic semisimple group and  $(G, H, \Gamma)$  a model set triple. Then every weakly admissible sequence  $(F_t)$  is a good approximation sequence for  $\Gamma$ .*

*Proof.* Every  $\omega \in S^+(G, K) \setminus \{1\}$  is a matrix coefficient for some non-trivial irreducible spherical representation  $V$  of  $G$ . Since  $V$  is irreducible and non-trivial, it does not contain any fixed vectors, hence its matrix coefficients decay by the ( $S$ -adic version of the) Howe–Moore theorem.  $\square$

Proposition 6.4 is related to the fact that the  $G$ -action on  $Y$  is *mixing*, a phenomenon which does not occur in the abelian case.

**6.3. Model sets in automorphism groups of trees.** Let  $T = T_d$  be a  $d$ -regular tree,  $o \in T$  a vertex,  $G := \text{Aut}(T)$  the full automorphism group and  $K := \text{Stab}(o)$ . Then  $K$  is a maximal compact subgroup of  $G$ , and  $(G, K)$  is a Gelfand pair (see [15] for a discussion of these Gelfand pairs).  $G$  contains a topologically simple subgroup  $G_0$  of index 2, which contains  $K$  (see [53]). If we fix a bi-partite coloring of the vertices of  $T$ , then  $G_0$  can be defined as the subgroup of coloring preserving automorphisms of  $T$ . By [34],  $G_0$  has the Howe–Moore property.

Let us briefly explain how to construct model sets in  $G_0$  using the results from [9, 10]. Given  $d \in \mathbb{N}$  we denote by  $S_d$  the permutation group on  $d$  letters and let  $F < S_d$  be a 2-transitive subgroup, e.g. the alternating group  $A_d$ . As explained in [9, Sec. 3.2] one obtains a subgroup  $U(F) < \text{Aut}(T_d)$  called the *universal group* of  $F$ . Roughly speaking,  $U(F)$  consists of those automorphisms of  $T_d$  which on each vertex link induce a permutation from  $F$  with respect to some fixed legal coloring, and different colorings lead to conjugate subgroups of  $\text{Aut}(T_d)$ . If  $d_1, d_2$  are large even integers ( $d_1 \geq 30$  and  $d_2 \geq 38$  is sufficient), then Burger and Mozes construct in [10, Thm. 6.3] a lattice

$$\Gamma_1 < U(A_{d_1}) \times U(A_{d_2}) < \text{Aut}(T_{d_1}) \times \text{Aut}(T_{d_2})$$

with the following properties. The projections of  $\Gamma_1$  into  $U(A_{d_1})$  and  $U(A_{d_2})$  are dense,  $\Gamma_1$  is torsion-free, cocompact and admits a finite index simple subgroup  $\Gamma$ .

**Proposition 6.5** (Model sets of Burger–Mozes type). *Let  $d \geq 30$  be an even integer,  $G := \text{Aut}(T_d)$ ,  $G_0 < G$  the index 2 subgroup,  $H := U(A_{38})$  and  $\Gamma$  as above. Then  $\Gamma$  is a simple cocompact lattice in  $G_0 \times H$ , and  $(G, H, \Gamma)$  and  $(G_0, H, \Gamma)$  are model set triples.*

*Proof.* By [10, Thm. 6.3],  $\Gamma_1$  is cocompact and projects densely into  $H$ ; the finite index subgroup  $\Gamma$  inherits these properties. The projection of  $\Gamma$  to  $G$  is non-trivial, since it has finite index in the projection of  $\Gamma_1$  which is dense in  $U(A_{d_1})$ . Since  $\Gamma$  is simple, the projection is injective and the finite index subgroup  $\Gamma \cap (G_0 \times H)$  coincides with  $\Gamma$ , i.e.  $\Gamma$  is contained in  $G_0 \times H$ .  $\square$

We now discuss approximation sequences for  $\Gamma$ . Observe that bi- $K$ -invariant subsets in  $G$  correspond via the projection  $\pi : G \rightarrow G/K = T$  to radial subsets of  $T$ . Thus the most obvious sequence of bi- $K$ -invariant sets in  $G$  is  $F_t := \pi^{-1}(B_t(o))$ , and these are easily seen to be weakly admissible. Similarly, the sets  $(F_{t,0}) := (F_t \cap G_0)$  form a weakly admissible sequence in  $G_0$ . Since  $G_0$  has the Howe-Moore property we deduce as in the Lie group case that the sequence  $(F_{t,0})$  is a good approximation sequence for  $\Gamma$  with respect to  $(G_0, K)$ . On the other hand we have:

**Proposition 6.6** (Nevo–Stein phenomenon for model sets of Burger–Mozes type). *The sequence  $(F_t)$  is not a good approximation sequence for  $\Gamma$  with respect to  $(G, K)$ , but the sequence  $(F_{2t})$  is.*

The second part of the proposition is immediate from the Howe-Moore property of  $G_0$ . The reason for the failure of the spectral property in the former case is a lack of mixing:

**Lemma 6.7.** *The action of  $G$  on  $Y = (G \times H)/\Gamma$  is not mixing. In fact,  $G_0$  does not act ergodically.*

*Proof.* If  $G_0$  acted ergodically on  $Y$ , then  $\Gamma$  would have to act ergodically on  $(G/G_0) \times H$  by the duality principle. However, a finite index subgroup of  $\Gamma$  preserves the open subset  $\{e\} \times H \subset (G/G_0) \times H$ . Since  $\Gamma$  is simple and infinite, this finite index subgroup coincides with  $\Gamma$ , contradicting ergodicity.  $\square$

*Proof of Proposition 6.6.* The Hecke algebra  $\mathcal{H}(G, K)$  is generated by a single element  $\sigma_1$ , hence spherical functions  $\omega$  are determined by  $\widehat{\sigma_1}(\omega)$ . Explicitly,  $\sigma_1$  is the characteristic function of the clopen subset  $\pi^{-1}(S_1(o))$ , where  $S_1(o) \subset T$  denotes the 1-sphere around  $o$ . The sign function  $\omega_0 \in \mathcal{S}^+(G, K)$  is the unique positive definite spherical function with  $\widehat{\sigma_1}(\omega_0) = -1$ .

Let now  $G_0$  as in Lemma 6.7. We deduce from the proposition that there exists a non-zero function  $f_0 \in L^2_o(Y)^{G_0}$ . Since  $K < G_0$ , the Hecke algebra  $\mathcal{H}(G, K)$  acts on  $L^2(Y)^{G_0} = L^2((G/G_0) \times$

$H/\Gamma$ ), and  $\sigma_1 * \sigma_1$  acts trivially on  $L^2(Y)^{G_0}$ . Thus if we define  $f := f_0 - \sigma_1 * f_0$ , then  $\sigma_1 * f = -f$ , and thus the sign function  $\omega_0$  is contained in the spectrum of  $L^2_o((G \times H)/\Gamma)$ . Since  $\widehat{\beta}_t(\omega_0) \not\rightarrow 0$  for the sequence  $(\beta_t)$  associated to  $(F_t)$ , the sequence  $(F_t)$  is not a good approximation sequence for  $\Gamma$ .  $\square$

The proposition shows in particular that the spectral condition in the approximation theorem is not always satisfied automatically in the non-amenable case and can be rather subtle.

## APPENDIX A. SPACES OF SUBSETS OF LOCALLY COMPACT GROUPS

Throughout this appendix we fix a lsc group  $G$ . We then denote by  $\mathcal{C}(G)$ ,  $\mathcal{O}(G)$  and  $\mathcal{K}(G)$  the sets of closed, open and compact subsets of  $G$  respectively. We also denote by  $e$  the identity element of  $G$  and by  $\mathfrak{U}(G) = \mathfrak{U}_e(G)$  the identity neighbourhood filter of  $G$ . We stress that elements of  $\mathfrak{U}(G)$  are identity neighbourhoods, but not necessarily open. The goal of this appendix is to describe various topologies on  $\mathcal{C}(G)$  and the corresponding orbit closures for subsets of  $G$  of finite local complexity. All the results presented in this appendix are well-known in the abelian case [41, 49, 3] and the generalizations to non-abelian groups discussed here are entirely routine. In the non-abelian case, the only treatment we are aware of is [56], which however focuses on different aspects.

**A.1. The Chabauty-Fell topology.** Given  $V \in \mathcal{O}(G)$  and  $K \in \mathcal{K}(G)$  we define subsets  $U_V, U^K \subset \mathcal{C}(G)$  by

$$U_V = \{C \in \mathcal{C}(G) \mid C \cap V \neq \emptyset\} \quad \text{and} \quad U^K = \{C \in \mathcal{C}(G) \mid C \cap K = \emptyset\}.$$

The topology on  $\mathcal{C}(G)$  generated by  $\{U_V\}_{V \in \mathcal{O}(G)} \cup \{U^K\}_{K \in \mathcal{K}(G)}$  is called the *Chabauty-Fell topology* on  $\mathcal{C}(G)$ . The following basic properties are well-known.

**Proposition A.1.** (i) *The Chabauty-Fell topology is a compact Hausdorff topology on  $\mathcal{C}(G)$ .*  
(ii) *The left and right-translation actions of  $G$  on  $\mathcal{C}(G)$  are jointly continuous.*  $\square$

Given  $P \in \mathcal{C}(G)$ ,  $K \in \mathcal{K}(G)$  and  $V \in \mathfrak{U}(G)$  we define

$$\widehat{U}_{K,V}(P) := \{Q \in \mathcal{C}(G) \mid Q \cap K \subset VP \text{ and } P \cap K \subset VQ\}.$$

**Proposition A.2.** *The sets  $\{\widehat{U}_{K,V}(P) \mid K \in \mathcal{K}(G), V \in \mathfrak{U}(G)\}$  generate the neighbourhood filter of  $P$  in the Chabauty-Fell topology.*

*Proof.* Denote by  $\tau$  the topology with neighbourhood filters given by the  $\{\widehat{U}_{K,V}(P)\}$ . We first show that every non-empty Chabauty-Fell open set  $U$  contains a non-empty  $\tau$ -open subset. We may assume that  $U$  is of the form

$$U = U^K \cap \bigcap_{i=1}^n U_{V_i},$$

$V_i \in \mathcal{O}(G)$  for all  $i \in \{1, \dots, n\}$  and  $K \in \mathcal{K}(G)$ . Since  $U \neq \emptyset$  we have  $V_i \setminus K \neq \emptyset$  for all  $i \in I$  and hence we find  $x_1, \dots, x_n \in G$  and  $W = W^{-1} \in \mathfrak{U}(G)$  such that

$$\overline{W}x_i \subset V_i \setminus K.$$

Let  $P := \{x_1, \dots, x_n\}$  and  $K' := K \cup \overline{W}P$  and note that the latter union is disjoint. We claim that  $\widehat{U}_{K',W}(P) \subset U$ . Indeed, let  $Q \in \widehat{U}_{K',W}(P)$  so that

$$Q \cap K' \subset WP \quad \text{and} \quad P \cap K' \subset WQ.$$

Then  $Q \cap K' \subset \overline{W}P = K' \setminus K$  and thus  $Q \cap K = \emptyset$ , i.e.  $Q \in U^K$ . Given  $i \in \{1, \dots, n\}$  we have  $x_i \in P = P \cap K' \subset WQ$  and hence  $Wx_i \cap Q = W^{-1}x_i \cap Q \neq \emptyset$ . We deduce that  $Q \cap V_i \supset Q \cap Wx_i \neq \emptyset$  and thus  $Q \in U_{V_i}$ . This shows that  $Q \in U$  and shows that  $\tau$  is finer than the Chabauty-Fell topology.

Conversely let  $P \in \mathcal{C}(G)$ ,  $K \in \mathcal{K}(G)$  and  $V \in \mathfrak{U}(G)$ . We construct a Chabauty-Fell open subset  $U$  of  $\widehat{U}_{K,V}(P)$  as follows. Firstly, let  $W \in \mathfrak{U}(G)$  be open and symmetric with  $W^2 \subset V$ . Secondly, let  $K' := K \setminus WP \in \mathcal{K}(G)$ . Since  $K \cap P$  is compact there exist  $t_1, \dots, t_n \in G$  such that

$$K \cap P \subset \bigcup_{i=1}^n Wt_i \quad \text{and} \quad (K \cap P) \cap Wt_i \neq \emptyset. \quad (\text{A.1})$$

We claim that

$$U := U^{K'} \cap \bigcap U_{Wt_i} \subset \widehat{U}_{K,V}(P).$$

Indeed, let  $Q \in U$ . Since  $Q \in U^{K'}$  we have  $\emptyset = Q \cap K' = Q \cap (K \setminus WP)$ , hence  $Q \cap K \subset WP \subset VP$ . Concerning the other inclusion, note that for  $i = 1, \dots, n$  we have  $Q \cap Wt_i \neq \emptyset$ , say  $q = wt_i$  with  $q \in Q$  and  $w \in W$ . Then  $Wt_i = Ww^{-1}q \in WW^{-1}Q \subset VQ$  and thus  $K \cap P \subset VQ$  by (A.1), finishing the proof.  $\square$

In the abelian context, this model for the Chabauty-Fell topology appears in [3], where it is referred to as the *local rubber topology*.

**A.2. Discrete subsets of locally compact groups.** We are going to consider various subspaces of  $\mathcal{C}(G)$  consisting of certain closed and discrete subsets of  $G$ . We will use the following language.

**Definition A.3.** Let  $G$  be a lcsc group and  $P \subset G$  be a subset.

- (1)  $P$  is called *locally finite* if it is closed and discrete.
- (2)  $P$  is called *uniformly locally finite* if for some (hence any) compact subset  $K \subset G$  there exists  $C > 0$  such that  $|P \cap gK| < C$  for all  $g \in G$ . (We then say that  $P$  is  $(K, C)$ -*locally finite*.)
- (3)  $P$  is called *uniformly discrete* if there exists an open subset  $V \subset G$  such that  $|P \cap gV| \leq 1$  for all  $g \in G$ . (We then say that  $P$  is  $V$ -*discrete*.)
- (4)  $P$  has *finite local complexity* if  $P^{-1}P$  is locally finite.

**Remark A.4.** (1) Any lcsc group  $G$  admits a left-invariant metric defining the topology, and the above properties can be expressed in terms of any such metric. This leads to the more classical definitions used e.g. in [1].

- (2) For  $P \subset G$  we have the obvious implications

$$P \text{ is uniformly discrete} \implies P \text{ is uniformly locally finite} \implies P \text{ is locally finite}$$

and none of these implications can be reversed. We will see in Lemma A.13 that every set of finite local complexity is uniformly discrete.

- (3) The collection  $\mathcal{D}(G)$  of locally finite subsets of  $G$  is a subset of  $\mathcal{C}(G)$  and thus inherits a subspace topology from the Chabauty-Fell topology. It is however not closed, hence the orbit closure of a locally finite subset of  $G$  may contain non-discrete subsets. We will see in Lemma A.12 below that if  $P_0 \subset G$  is of finite local complexity, then its orbit closure consists entirely of sets of finite local complexity, hence it is contained in  $\mathcal{D}(G)$ .

**A.3. The local uniformity and the local topology.** Let  $G$  be a lcsc group. We are going to define a  $G$ -invariant uniformity on  $\mathcal{C}(G)$  whose associated topology is finer than the Chabauty-Fell topology, but coincides with the Chabauty-Fell topology on the orbit closure of any set of finite local complexity. For every  $K \in \mathcal{K}(G)$  and every  $V \in \mathcal{U}(G)$  we define a subset  $U_{K,V} \subset \mathcal{C}(G) \times \mathcal{C}(G)$  by

$$U_{K,V} := \{(P, Q) \in \mathcal{C}(G) \mid \exists t \in V : P \cap K = tQ \cap K\}. \quad (\text{A.2})$$

**Proposition A.5.** The set  $\mathcal{B} := \{U_{K,V} \mid K \in \mathcal{K}(G), V \in \mathcal{U}(G)\}$  is a fundamental system of entourages for a uniformity on  $\mathcal{C}(G)$ .

*Proof.* In the notation of [8, Chapter 2, § 1.1] we have to show that

- (B1) the diagonal  $\Delta(\mathcal{C}(G))$  is contained in every  $U \in \mathcal{B}$ ;
- (B2) for all  $U_1, U_2 \in \mathcal{B}$  there exists  $U_3 \in \mathcal{B}$  with  $U_3 \subset U_1 \cap U_2$ ;
- (B3) for all  $U_1 \in \mathcal{B}$  there exists  $U_2 \in \mathcal{B}$  such that  $U_2 \subset U_1^{-1}$ ;
- (B4) for all  $U_1 \in \mathcal{B}$  there exists  $U_2 \in \mathcal{B}$  such that  $U_2^2 \subset U_1$ .

We establish (B1) – (B4) for our  $\mathcal{B}$  at hand.

(B1) is immediate from the fact that  $e \in V$  for every  $V \in \mathfrak{U}(G)$ .

(B2) If  $(P, Q) \in U_{K_1 \cup K_2, V_1 \cap V_2}$ , then there exists  $t \in V_1 \cap V_2$  such that

$$P \cap (K_1 \cup K_2) = tQ \cap (K_1 \cup K_2).$$

It follows that for  $j = 1, 2$ ,

$$P \cap K_j = tQ \cap K_j,$$

whence  $(P, Q) \in U_{K_1, V_1} \cap U_{K_2, V_2}$ . This shows that  $U_{K_1 \cup K_2, V_1 \cap V_2} \subset U_{K_1, V_1} \cap U_{K_2, V_2}$ .

(B3) Let  $V \in \mathfrak{U}(G)$  and  $K \in \mathcal{K}(G)$ . There exists a compact  $W \in \mathfrak{U}(G)$  with  $W^2 \subset V$  and  $W = W^{-1}$ . Then  $WK$  is compact and if  $(P, Q) \in U_{WK, W}$ , then there exists  $t \in W$  such that

$$P \cap WK = tQ \cap WK.$$

By assumption,  $s := t^{-1} \in W \subset V$  and  $e \in sW$ , hence  $K \subset sWK$ . We obtain

$$Q \cap K = (Q \cap sWK) \cap K = (sP \cap sWK) \cap K = sP \cap K,$$

showing that  $(Q, P) \in U_{K, V}$ , whence  $U_{WK, W} \subset U_{K, V}^{-1}$ .

(B4) Let  $V, K, W$  as in the proof of (B3) and let  $(P, R) \in U_{WK, W}^2$ . Then there exist  $Q \in \mathcal{D}(G)$  such that  $\{(P, Q), (Q, R)\} \subset U_{WK, W}$ , i.e. there exist  $t_1, t_2 \in W$  such that

$$P \cap WK = t_1 Q \cap WK, \quad Q \cap WK = t_2 R \cap WK.$$

Let  $t := t_1 t_2 \in V$ . Since  $K \subset t_1 WK \cap WK$  we then have

$$\begin{aligned} tR \cap K &= (t_1 t_2 R \cap t_1 WK) \cap K = t_1 (t_2 R \cap WK) \cap K = t_1 (Q \cap WK) \cap K \\ &= t_1 Q \cap K = (t_1 Q \cap WK) \cap K = (P \cap WK) \cap K = P \cap K, \end{aligned}$$

hence  $(P, R) \in U_{V, K}$ , showing that  $U_{WK, W}^2 \subset U_{K, V}$ .

This establishes (B1) – (B4) and finishes the proof.  $\square$

In the sequel we refer to the uniformity defined in Proposition A.5 as the *local uniformity* on  $\mathcal{C}(G)$ . and the corresponding topology as the *local topology*. By definition, a neighbourhood basis of  $P \in \mathcal{C}(G)$  in the local topology is given by the sets

$$\begin{aligned} U_{K, V}(P) &= \{Q \in \mathcal{C}(G) \mid (P, Q) \in U_{K, V}\} \\ &= \{Q \in \mathcal{C}(G) \mid \exists t \in V : tQ \cap K = P \cap K\}. \end{aligned}$$

where  $K$  runs through  $\mathcal{K}(G)$  and  $V$  runs through  $\mathfrak{U}(G)$

**Lemma A.6.** *The  $G$ -action on  $\mathcal{C}(G)$  by left-translations is jointly continuous with respect to the local topology.*

*Proof.* Let us denote by  $m : G \times \mathcal{C}(G) \rightarrow \mathcal{C}(G)$  the left-translation action of  $G$ , and let  $g \in G$ ,  $P \in \mathcal{C}(G)$ . We are going to show continuity of  $m$  at  $(g, P)$ . For this let  $K \subset G$  be compact and  $V \subset G$  be an open identity neighbourhood. We choose a symmetric identity neighbourhood  $W$  with  $W^2 \subset V$  and define  $K' := g^{-1}K$  and  $V' := g^{-1}Wg$ . Now let  $h \in Wg$  and  $Q \in U_{K', V'}(P)$ . We then find  $s \in V'$  such that  $sQ \cap K' = P \cap K'$  and thus

$$(gsh^{-1})(hQ) \cap K = gP \cap K.$$

Since  $s \in V'$  we have  $gsg^{-1} \in W$  and since also  $gh^{-1} \in W$  we obtain

$$t := gsh^{-1} = (gsg^{-1})(gh^{-1}) \in W^2 \subset V.$$

To summarize, we have found  $t \in V$  such that

$$t(hQ) \cap K = gP \cap K.$$

This shows that  $hQ \in U_{K, V}(gP)$  and thus  $Wg \times U_{K', V'}(P) \subset m^{-1}(U_{K, V}(gP))$ , which implies continuity of  $m$  at  $(g, P)$ .  $\square$

For the following proposition we denote by  $\tau_{CF}$  the Chabauty-Fell topology on  $\mathcal{C}(G)$  and by  $\tau_{loc}$  the local topology.

**Proposition A.7.** *The identity map  $(\mathcal{C}(G), \tau_{loc}) \rightarrow (\mathcal{C}(G), \tau_{CF})$  is continuous, i.e. the local topology is finer than the Chabauty topology (hence in particular Hausdorff).*

*Proof.* We show that every  $\tau_{CF}$ -neighbourhood  $\hat{U}$  of  $P \in \mathcal{C}(G)$  contains a  $\tau_{loc}$ -neighbourhood of  $P$ . By Proposition A.2 we may assume that  $\hat{U} = \hat{U}_{K,V}(P)$  for some  $K \in \mathcal{K}(G)$ ,  $V \in \mathcal{U}(G)$ . Let  $K'$  be a compact set such that

$$K \subset \bigcap_{t \in V} t^{-1}K'$$

and let  $V' \in \mathcal{U}(G)$  such that  $(V')^{-1}V' \subset V$ . We claim that  $U_{K',V'}(P) \subset \hat{U}_{K,V}(P)$ . Indeed, let  $Q \in U_{K',V'}(P)$  and let  $t \in V'$  such that  $tQ \cap K' = P \cap K'$ . Then  $P \cap K \subset V'Q' \subset VQ$  and moreover

$$Q \cap t^{-1}K' = t^{-1}P \cap t^{-1}K' \Rightarrow Q \cap K = t^{-1}P \cap K \subset (V')^{-1}P \cap K \subset VP \cap K.$$

This shows that  $Q \in \hat{U}_{K,V}(P)$  and finishes the proof.  $\square$

For the following proposition we recall that we denote by  $\mathcal{D}(G) \subset \mathcal{C}(G)$  the subset of locally finite subsets of  $G$ .

**Proposition A.8.** *The restriction of the local uniformity to  $\mathcal{D}(G)$  is complete.*

*Proof.* Let  $(I, \leq)$  be a directed set and  $(P_i)_{i \in I}$  be a Cauchy net in  $\mathcal{D}(G)$  with respect to the local uniformity. We have to show that  $(P_i)_{i \in I}$  admits a convergent subnet. Either there exists a subnet converging to the empty set, or after passing to a subnet we may assume that there exists  $K \in \mathcal{K}(G)$  such that  $P_i \cap K \neq \emptyset$  for all  $i \in I$ . In the latter case we can choose  $t_i \in P_i \cap K$  and passing to another subnet we may assume that  $t_i \rightarrow t$ . Then  $(t_i^{-1}P_i)$  is again a Cauchy net, and convergence of  $(P_i)$  is equivalent to convergence of  $(t_i^{-1}P_i)$ . We may thus assume that  $e \in P_i$  for every  $i \in I$ .

Now let  $V_0$  be a compact identity neighbourhood. Since  $(P_i)$  is a Cauchy net there exists  $i_0 \in I$  such that for every  $i \geq i_0$  we have  $(P_{i_0}, P_i) \in U_{V_0, V_0}$ . Thus for every  $i \geq i_0$  there exists  $s_i \in V_0$  such that

$$s_i P_{i_0} \cap V_0 = P_i \cap V_0.$$

Since  $e \in P_i \cap V_0 \subset s_i P_{i_0}$  we have  $s_i \in P_{i_0}^{-1}$ . Since also  $s_i \in V_0$ , we deduce that for all  $i \geq i_0$ ,

$$P_i \cap V_0 = s_i P_{i_0} \cap V_0 = s_i (P_{i_0} \cap s_i^{-1} V_0) \subset (P_{i_0}^{-1} \cap V_0) (P_{i_0} \cap V_0^{-1} V_0). \quad (\text{A.3})$$

Now  $P_{i_0}$  and hence  $P_{i_0}^{-1}$  are locally finite, and  $V_0$  and hence  $V_0^{-1}V_0$  are compact. It follows that

$$\mathcal{F} := (P_{i_0}^{-1} \cap V_0) (P_{i_0} \cap V_0^{-1} V_0)$$

is finite, and hence  $V_1 := (V_0 \setminus \mathcal{F}) \cup \{e\}$  is an identity neighbourhood. By (A.3) we have

$$P_i \cap V_1 = \{e\}$$

for all  $i \geq i_0$ .

Now let  $K \in \mathcal{K}(G)$  be arbitrary. Since  $(P_i)$  is a Cauchy filter we find  $i_1 \geq i_0$  such that for all  $l, m \geq i_1$  we have  $(P_l, P_m) \in U_{K, V_1}$ . Thus there exists  $t \in V_1$  such that

$$tP_l \cap K = P_m \cap K.$$

Now assume  $K \supset V_1$ . Since  $e \in P_l$  we have  $t \in tP_l \cap K$ , and hence  $t \in P_m \cap K$ . In particular,  $t \in P_m \cap V_1 = \{e\}$  and thus  $P_l \cap K = P_m \cap K$ . To summarize, for every sufficiently large  $K$  (and hence, a posteriori, for every  $K$ ), there exists  $i_K \in I$  such that for all  $j \geq i_K$  we have  $P_j \cap K = P_{i_K} \cap K$ . We may assume that  $i_K \leq i_{K'}$  whenever  $K \subset K'$ . In particular, if we define

$$P := \bigcup_{K \in \mathcal{K}(G)} P_{i_K} \cap K$$

then for all  $j \geq i_K$  we have

$$P_j \cap K = P \cap K.$$

This shows both that  $P$  is locally finite, since  $P \cap K$  is finite for every  $K$ , and that  $(P_i)$  converges to  $P$ .  $\square$

From this completeness property we derive the following compactness criterion.

**Corollary A.9.** *A subset  $C \subset \mathcal{D}(G)$  is pre-compact with respect to the local topology if and only if for every  $K \in \mathcal{K}(G)$  and  $U \in \mathcal{U}(G)$  there exists a finite subset  $\mathcal{F} \subset \mathcal{D}(G)$  such that*

$$C \subset \bigcup_{P \in \mathcal{F}} U_{K,V}^{-1}(P).$$

*Proof.* Since the local topology on  $\mathcal{D}(G)$  is complete, it follows from [8, Chapter 2, § 4.2, Theorem 3] that a subset  $C \subset \mathcal{D}(G)$  is pre-compact if and only if for every  $K \in \mathcal{K}(G)$  and  $V \in \mathcal{U}(G)$  there exists a finite subset  $\mathcal{F} \subset \mathcal{D}(G)$  such that

$$C \subset \bigcup_{P \in \mathcal{F}} U_{K,V}(P).$$

By axiom (B3) in the proof of Proposition A.5 we can replace  $U_{K,V}$  by  $U_{K,V}^{-1}$ .  $\square$

**A.4. Orbit closures of sets of finite local complexity.** It turns out that finite local complexity can be characterized as a compactness property as follows.

**Theorem A.10.** *Let  $P \in \mathcal{D}(G)$ . Then the following are equivalent.*

- (i)  *$P$  has finite local complexity, i.e.  $P^{-1}P \subset G$  is locally finite.*
- (ii) *The  $G$ -orbit  $G.P \subset \mathcal{C}(G)$  is pre-compact with respect to the local topology.*
- (iii)  *$\forall K \in \mathcal{K}(G) \exists K' \in \mathcal{K}'(G) \forall t \in G \exists t' \in K' : tP \cap K = t'P \cap K$ .*

*Proof.* (i)  $\Rightarrow$  (ii): Assume that  $P^{-1}P$  is locally finite and fix  $K \in \mathcal{K}(G)$  and  $V \in \mathcal{U}(G)$ . Then the intersection  $F := P^{-1}P \cap K^{-1}K$  is finite since  $K^{-1}K$  is compact. Moreover, finitely many right- $V$ -translates cover  $K$ , i.e. there exists another finite set  $E$  such that  $K \subset VE$ . We claim that for all  $g \in G$  we get

$$gP \in \bigcup_{P' \in \mathcal{F}} U_{K,V}^{-1}(P'), \tag{A.4}$$

where

$$\mathcal{F} := \{sF' \in \mathcal{D}(G) \mid F' \subseteq F, s \in E\}.$$

Since  $\mathcal{F} \subset \mathcal{D}(G)$  is finite, this will imply pre-compactness of  $G.P$  by Corollary A.9. Thus it remains only to show (A.4).

If  $gP \cap K = \emptyset$  then there is nothing to show. Otherwise we can choose  $p \in P$  such that  $gp \in K \subset VE$ . Then  $F' := p^{-1}P \cap (gp)^{-1}K \subseteq F$  and we find  $s \in E$  and  $v \in V$  such that  $gp = vs$ . We then compute

$$gP \cap K = gp(p^{-1}P \cap (gp)^{-1}K) \cap K = vs(p^{-1}P \cap (gp)^{-1}K) \cap K = vsF' \cap K,$$

which shows that  $gP \in U_{K,V}^{-1}(sF')$  and proves (A.4).

(ii)  $\Rightarrow$  (iii): If the orbit  $G.P$  is precompact, then for every  $K \in \mathcal{K}(G)$  and  $V \in \mathcal{U}(G)$  the open cover

$$G.P \subset \bigcup_{g \in G} U_{K,V}^{-1}(g.P)$$

has a finite subcover. Given  $K \in \mathcal{K}(G)$  we can thus choose a compact  $V \in \mathcal{U}(G)$  and  $t_1, \dots, t_n \in G$  such that

$$G.P \subset \bigcup_{i=1}^n U_{K,V}^{-1}(t_i P).$$



Set  $K' := \bigcup V t_i$ , then for every  $t \in G$  there exists  $s \in V$  and  $i \in \{1, \dots, n\}$  such that

$$tP \cap K = st_i P \cap K.$$

Hence if we define  $t' := st_i$ , then  $t \in K'$  and  $tP \cap K = t'P \cap K$ .

(iii)  $\Rightarrow$  (i): Given  $P$  satisfying (iii) we will show that  $P^{-1}P \cap K$  is finite for every  $K \in \mathcal{K}(G)$ . We may assume that  $e \in K$  and choose  $K' \in \mathcal{K}(G)$  as in (iii). We will show that

$$P^{-1}P \cap K \subset (P^{-1} \cap K')(P \cap (K')^{-1}K), \quad (\text{A.5})$$

which is finite. Thus let  $q \in P^{-1}P \cap K$  and choose  $q_1, q_2 \in P$  with  $q = q_2^{-1}q_1$ . By assumption there exists  $t' \in K'$  such that

$$q_2^{-1}P \cap K = t'P \cap K \subset t'P.$$

We have  $e \in q_2^{-1}P \cap K$ , hence  $e \in t'P$ , i.e.  $t' \in P^{-1} \cap K'$ . Thus

$$q = q_2^{-1}q_1 \in q_2^{-1}P \cap K = t'P \cap K = t'(P \cap (t')^{-1}K) \subset (P^{-1} \cap K')(P \cap (K')^{-1}K),$$

which establishes (A.5) and finishes the proof.  $\square$

**Corollary A.11.** *Let  $P \in \mathcal{D}(G)$  and  $\widehat{X} := \overline{G \cdot P} \subset \mathcal{C}(G)$  its orbit closure in the local topology. Then the Chabauty-Fell topology and the local topology coincide on  $\widehat{X}$  if and only if  $P$  is of finite local complexity.*

*Proof.* If  $P$  has finite local complexity, then  $(\widehat{X}, \tau_{\text{loc}})$  is compact and hence the continuous map  $(\widehat{X}, \tau_{\text{loc}}) \rightarrow (\widehat{X}, \tau_{CF})$  is a homeomorphism. Conversely, if the topologies coincide, then  $(\widehat{X}, \tau_{\text{loc}})$  is compact, hence  $P$  has finite local complexity.  $\square$

From now on let  $P_0 \subset G$  be a subset of finite local complexity and consider the orbit closure

$$\widehat{X} := \widehat{X}_{P_0} := \overline{G \cdot P_0} \subset (\mathcal{C}(G), \tau_{\text{loc}}).$$

By Corollary A.11 the Chabauty-Fell topology and the local topology coincide on  $\widehat{X}$ , and in particular  $\widehat{X}$  is the orbit closure of  $P_0$  with respect to either of these topologies.

**Lemma A.12.** *Assume that  $P_0$  has finite local complexity. Then for every  $P \in \widehat{X}_{P_0}$  we have*

$$P^{-1}P \subset P_0^{-1}P_0.$$

*In particular, every such  $P$  has finite local complexity.*

*Proof.* Since  $P \in \widehat{X}$  there exist  $g_n \in G$  such that  $g_n P_0 \rightarrow P$  in the local topology, whence for every  $i \in \mathbb{N}$  there exists  $n_i \in \mathbb{N}$  and  $t_i \in V_i$  such that

$$t_i g_{n_i} P_0 \cap K_i = P \cap K_i.$$

Then

$$(P \cap K_i)^{-1}(P \cap K_i) \subset (t_i g_{n_i} P_0)^{-1}(t_i g_{n_i} P_0) = P_0^{-1}P_0.$$

$\square$

**Lemma A.13.** *If  $P \in \mathcal{D}(G)$  is of finite local complexity, then it is uniformly discrete (and hence uniformly locally finite).*

*Proof.* Assume  $P$  is not uniformly discrete and let  $V_n \in \mathcal{U}(G)$  be a sequence converging to  $e$ . For every  $n \in \mathbb{N}$  we then find  $g_n \in G$  such that  $|g_n V_n \cap P| \geq 2$ . If  $x_n$  and  $y_n$  are two distinct points in  $g_n V_n \cap P$ , then  $p_n := x_n^{-1}y_n \in V_n^{-1}V_n \cap P^{-1}P$  and  $p_n \neq e$ . In particular  $p_n \in P^{-1}P \setminus \{e\}$  and  $p_n \rightarrow e$ , so  $P^{-1}P$  is not locally finite.  $\square$

If  $P_0$  has finite local complexity, then every  $P \in \widehat{X}_{P_0}$  has finite local complexity by Lemma A.12, hence is also uniformly locally finite by the previous lemma. In fact, the finiteness parameter is even uniform over the orbit closure.

**Lemma A.14.** *Let  $K \in \mathcal{K}(G)$  and assume that  $P_0 \subset G$  has finite local complexity. Then there exists a uniform constant  $C_K > 0$  such that every  $P \in \widehat{X}_{P_0}$  is  $(K, C_K)$ -locally finite.*

*Proof.* Fix  $K \in \mathcal{K}(G)$ . By Lemma A.13 there exists  $C = C_K > 0$  such that  $|P_0 \cap gK| \leq C$  for all  $g \in G$ . Now let  $P \in \widehat{X} = \widehat{X}_{P_0}$ . By Corollary A.11,  $\widehat{X}$  coincides with the orbit closure of  $P_0$  in the left-local topology. Thus for every compact set  $L$ , some translate of  $P_0$  is contained in  $U_{L,G}(P)$ , i.e. there exists  $t \in G$  such that

$$tP_0 \cap L = P \cap L.$$

This implies that for every  $g \in G$ ,

$$|P \cap gK \cap L| = |tP_0 \cap L \cap gK| \leq |tP_0 \cap gK| = |P_0 \cap t^{-1}gK| \leq C.$$

Since  $L$  was arbitrary this implies  $|P \cap gK| \leq C$ .  $\square$

**A.5. Cut-and-project sets have finite local complexity.** Let  $G, H$  be lsc groups. We denote by  $\pi_G : G \times H \rightarrow G$ , respectively  $\pi_H : G \times H \rightarrow H$  the canonical projections. Let  $\Gamma \subset G \times H$  be a discrete (hence locally finite) subgroup and  $W_0 \subset H$  be a compact set. Then the *cut-and-project set* associated with the pair  $(W_0, \Gamma)$  is defined as

$$P_0 := P_0(W_0, \Gamma) := \pi_G((G \times W_0) \cap \Gamma).$$

**Lemma A.15.** *If  $P_0 = P_0(W_0, \Gamma)$  is a cut-and-project set, then both  $P_0$  and  $P_0^{-1}P_0$  are of finite local complexity, hence uniformly discrete.*

*Proof.* Firstly, we show that  $P_0$  is locally finite. For this we observe that if  $K \in \mathcal{K}(G)$ , then

$$P_0 \cap K \subset \pi_G((K \times W_0) \cap \Gamma),$$

and since  $\Gamma$  is locally finite, the sets  $(K \times W_0) \cap \Gamma$  and thus also  $P_0 \cap K$  are finite.

Secondly, we observe that  $P_0 = \pi_G((G \times W_0) \cap \Gamma)$  and thus  $P_0^{-1}P_0 \subset P'_0 := \pi_G((G \times W_0^{-1}W_0) \cap \Gamma)$  is a subset of another cut-and-project set.

By the first observation (applied to  $W_0^{-1}W_0$ ) the set  $P'_0$  is locally finite, hence by the second observation  $P_0^{-1}P_0$  is locally finite, i.e.  $P_0$  has finite local complexity. By the same argument applied to  $W_0^{-1}W_0$  we see that  $P'_0$  has finite local complexity, and hence  $P_0^{-1}P_0$  has finite local complexity by the second observation.  $\square$

#### A.6. Delone sets and Meyer sets.

**Definition A.16.** Let  $P \subset G$  be a subset.

- (1)  $P$  is called *left-syndetic* if there exists a compact subset  $K \subset G$  such that  $G = PK$ .
- (2)  $P$  is called a *Delone set* if it is both left-syndetic and uniformly discrete.
- (3)  $P$  is called a *Meyer set* if it is a Delone set and  $P^{-1}P$  is uniformly discrete.

Note that for non compactly generated locally compact abelian groups, our definition of Meyer set is different from the notion used e.g. in [52].

**Lemma A.17.** *Let  $P_0 \in \mathcal{D}(G)$ . Then  $P_0$  is left-syndetic if and only if the empty set is not contained in the orbit closure  $\widehat{X}_{\text{loc}}$  of  $P_0$  with respect to the local topology.*

*Proof.* Assume that  $\emptyset \in \widehat{X}_{\text{loc}}$  and let  $K \in \mathcal{K}(G)$  and  $V \in \mathcal{U}(G)$ . Then there exists an element  $g \in G$  such that  $gP_0 \in U_{K^{-1},V}(\emptyset)$ , whence there exists  $t \in V$  such that  $tgP_0 \cap K^{-1} = \emptyset$  and thus

$$g^{-1}t^{-1} \notin P_0K.$$

Consequently  $G \neq P_0K$  and since  $K \in \mathcal{K}(G)$  was arbitrary,  $P_0$  is not left-syndetic.

Conversely assume that  $P_0$  is not left-syndetic. Then for every compact set  $K^{-1}$  the set  $P_0K^{-1}$  is a proper subset of  $G$ . It is moreover closed in  $G$  as a product of a closed and a compact set. Thus for every  $g_K \in G \setminus P_0K^{-1}$  there exists a symmetric open identity neighbourhood  $V_K$  such

that  $g_K^{-1}V_K \subset G \setminus P_0K^{-1}$  and thus  $g_K^{-1}V^{-1} \subset G \setminus P_0K^{-1}$  for every  $V \subset V_K$ . We deduce that for all  $K \in \mathcal{K}(G)$  and  $V \subset V_K$  there exists  $g_K \in G$  such that for all  $t \in V$ ,

$$g_K^{-1}t^{-1} \notin P_0K^{-1},$$

and thus

$$tg_KP_0 \cap K = \emptyset.$$

This shows in particular that  $g_KP_0 \in U_{K,V}(\emptyset)$ . Thus if  $(K_n)$  is an ascending union of compact sets and  $(V_n)$  is a descending union of identity neighbourhoods intersecting only in  $\{e\}$  with  $V_i \subset V_{K_i}$  then  $g_{K_n}P_0 \rightarrow \emptyset$ , whence  $\emptyset \in \widehat{X}_{\text{loc}}$ .  $\square$

**Corollary A.18.** *Let  $P_0 \subset G$  be a subset of finite local complexity. Then  $P_0$  is left-syndetic if and only if  $\emptyset \notin \widehat{X}_{P_0}$ .*  $\square$

By Lemma A.15, a cut-and-project set is Meyer if and only if it is relatively dense, which is not always the case. The following proposition provides some sufficient conditions.

**Proposition A.19.** *Let  $P_0 = P_0(W_0, \Gamma)$  be a cut-and-project set associated with a lattice  $\Gamma < G \times H$ . Assume that  $W_0$  has non-empty-interior, that  $\Gamma$  is cocompact,  $\pi_G|_\Gamma$  is injective and that  $\pi_H(\Gamma)$  is dense in  $H$ . Then  $P_0$  is relatively dense and thus a Meyer set and  $\emptyset \notin \widehat{X}_{P_0}$ .*

*Proof.* Since  $\Gamma$  projects injectively onto  $G$  we can define a map

$$\tau : \pi_G(\Gamma) \rightarrow H, \quad g \mapsto \pi_H \circ \pi_G|_\Gamma^{-1}(g).$$

Note that  $P_0 = \tau^{-1}(W)$ . Since  $\Gamma$  is co-compact in  $G \times H$ , there exist compact sets  $K \subset G$  and  $L \subset G$  such that  $(K \times L)\Gamma = G \times H$ . In particular, for every  $g \in G$ , we have

$$\Gamma \cap (K^{-1}g \times L^{-1}) \neq \emptyset.$$

Since  $\pi_G|_\Gamma$  is injective we thus have

$$\pi_G(\Gamma \cap (K^{-1}g \times H)) \cap \pi_G(\Gamma \cap (G \times L^{-1})) = \pi_G(\Gamma \cap (K^{-1}g \times L^{-1})) \neq \emptyset. \quad (\text{A.6})$$

Now let  $V$  denote the interior of  $W$ . Since  $V$  is assumed open and non-empty and  $\Gamma_H$  is dense, we have  $\Gamma_H V = H$ , and since  $L \subset H$  is compact, there exists a finite subset  $S \subset \Gamma_H$  such that  $L^{-1} \subset SV$ . In particular, since  $(\tau^{-1}(s), s) \in \Gamma$  for every  $s \in S$  we have

$$\pi_G(\Gamma \cap (G \times L^{-1})) \subset \pi_G(\Gamma \cap (G \times SV)) = \tau^{-1}(S) \cdot \pi_G(\Gamma \cap (G \times V)),$$

and combining this with (A.6) we find that

$$(\tau^{-1}(S))^{-1} \cdot \pi_G(\Gamma \cap (K^{-1}g \times H)) \cap \pi_G(\Gamma \cap (G \times V)) \neq \emptyset.$$

Since  $\pi_G(\Gamma \cap (G \times V)) = \tau^{-1}(V)$  this implies that for every  $g \in G$ ,

$$(\tau^{-1}(S))^{-1}K^{-1}g \cap \tau^{-1}(V) \neq \emptyset.$$

If we define  $F = K\tau^{-1}(S)$ , then we can rewrite this as  $G = F\tau^{-1}(V)$ , which in view of  $P_0 = \tau^{-1}(W) \supset \tau^{-1}(V)$  implies  $G = FP_0$ . Since  $S$  is finite and  $K$  is compact, we see that  $F$  is compact as well, hence  $P_0$  is right-syndetic. By Corollary A.18 it is also left-syndetic, hence relatively dense, hence Meyer by Proposition A.15. By Corollary A.18 this implies  $\emptyset \notin \widehat{X}_{P_0}$ .  $\square$

## APPENDIX B. GELFAND PAIRS AND SPHERICAL HARMONIC ANALYSIS

In this appendix we collect some standard facts concerning Gelfand pairs and their spherical Fourier transforms for ease of reference. Our basic references are [55, 17]. Let  $G$  be a lcsc group and  $K < G$  be a compact subgroup.

**B.1. Characterizations of Gelfand pairs.** According to [55, Thm. 9.8.1] the following properties of the pair  $(G, K)$  are equivalent:

- (1) The Hecke algebra  $\mathcal{H}(G, K) := C_c(K \backslash G / K)$  of bi- $K$ -invariant compactly supported continuous functions on  $G$  is commutative under convolution.
- (2) The Banach algebra  $L^1(K \backslash G / K)$  of bi- $K$ -invariant absolutely Haar-integrable function classes on  $G$  is commutative under convolution.
- (3) The measure algebra  $\mathcal{M}(K \backslash G / K)$  of finite bi- $K$ -invariant complex-valued measures on  $G$  is commutative under convolution.
- (4) The  $G$ -representation  $L^2(G / K)$  is multiplicity free.
- (5) If  $V$  is an irreducible unitary  $G$ -representation and  $V^K < V$  denotes the subspace of  $K$ -invariants, then  $\dim V^K \leq 1$ .

If the pair  $(G, K)$  satisfies these properties then it is called a *Gelfand pair*.

**B.2. Spherical representations.** Let  $(G, K)$  be a Gelfand pair. Every unitary representation  $(V, \pi)$  of  $G$  extends to a representation of the measure algebra  $\mathcal{M}(G)$  which we denote by the same letter  $\pi$ , and the action of the abelian subalgebra  $\mathcal{M}(K \backslash G / K)$  preserves the subspace  $V^K$  of  $K$ -invariant vectors. By restriction we also obtain representations of  $L^1(K \backslash G / K)$  and  $\mathcal{H}(G, K)$  on  $V^K$  by bounded linear operators. Explicitly the latter representation is given by

$$\pi : \mathcal{H}(G, K) \rightarrow \mathcal{B}(V^K), \quad \pi(f).v = \int_G f(g) \cdot (\pi(g).v) dm_G(g).$$

A  $G$ -representation  $(V, \pi_V)$  is called *spherical* if  $V^K \neq \{0\}$ . By characterization (5) of a Gelfand pair, if  $(V, \pi_V)$  is an irreducible spherical representation then  $\dim V^K = 1$  and thus there exists a unique up to sign unit vector  $v \in V^K$ . It follows that the bi- $K$ -invariant matrix coefficient

$$\omega_V : G \rightarrow \mathbb{C}, \quad \omega_V(g) := \langle v, \pi_V(g).v \rangle \tag{B.1}$$

depends only on the representation  $(V, \pi_V)$ .

**B.3. Spherical functions.** A continuous function  $\omega : G \rightarrow \mathbb{C}$  is called *spherical* (with respect to  $(G, K)$ ) if the associated Radon measure

$$m_\omega(f) := \int_G f(x) \omega(x^{-1}) dm_G(x)$$

is a *spherical measure*, i.e. if it is bi- $K$ -invariant and restricts to a (not necessarily continuous) algebra homomorphism  $m_\omega : \mathcal{H}(G, K) \rightarrow \mathbb{C}$ . Any spherical function is itself bi- $K$ -invariant and satisfies  $\omega(e) = 1$ . Moreover, it is a joint eigenfunction for the Hecke algebra, i.e. for every  $f \in \mathcal{H}(G, K)$  there exists  $\lambda_\omega(f) \in \mathbb{C}$  such that  $f * \omega = \lambda_\omega(f) \omega$  (see [55, Thm. 8.2.6]).

In the sequel we denote by  $\mathcal{S}(G, K)$  the space of spherical functions and by  $\mathcal{S}_b(G, K)$  the subset of bounded spherical functions. Given  $\omega \in \mathcal{S}_b(G, K)$  the linear functional  $m_\omega$  is  $L^1$ -continuous and thus extends to a linear functional on  $L^1(K \backslash G / K)$ . This provides an identification between  $\mathcal{S}_b(G, K)$  and the Gelfand spectrum of the Banach algebra  $L^1(K \backslash G / K)$  (see [55, Thm. 8.2.7]). In particular,  $\mathcal{S}_b(G, K)$  carries a natural locally compact Hausdorff topology given by pointwise convergence of the associated functionals on the Hecke algebra.

**B.4. Positive definite spherical functions.** A spherical function  $\omega$  is called *positive definite* if  $m_\omega(f^* * f) \geq 0$  for all  $f \in \mathcal{H}(G, K)$ , and we denote by  $\mathcal{S}^+(G, K)$  the set of positive definite spherical functions. Every positive definite spherical function is bounded, and the subspace  $\mathcal{S}^+(G, K) \subset \mathcal{S}_b(G, K)$  is closed, hence inherits a locally compact topology (see [55, Prop. 8.4.2 and Prop. 9.2.9]). We record the following properties of positive definite spherical functions for ease of reference; see [55, Thm. 8.2.6 and Prop. 8.4.8].

**Lemma B.1** (Properties of positive definite spherical functions). *Let  $\omega \in \mathcal{S}^+(G, K)$ .*

- (i)  $\omega^* = \omega$ .
- (ii) If  $\rho : G \rightarrow \mathbb{C}$  is a bi- $K$ -invariant continuous function which satisfies  $f * \rho = \widehat{f}(\omega) \cdot \rho$  for all  $f \in \mathcal{H}(G, K)$ , then  $\rho = \rho(e) \cdot \omega$ .  $\square$

Positive definite spherical functions correspond to spherical representations in the following sense. For every irreducible spherical representation  $(V, \pi_V)$  the associated canonical bi- $K$ -invariant matrix coefficient  $\omega_V$  as defined in (B.1) is a positive definite spherical function, and the map  $(V, \pi_V) \mapsto \omega_V$  defines a bijection between unitary equivalence classes of irreducible spherical representations and positive definite spherical functions [55, Thm. 8.4.8]. In the sequel we denote the inverse of this bijection by  $\omega \mapsto (V_\omega, \pi_{V_\omega})$ .

**B.5. The spherical Fourier transform and the Plancherel measure.** We can now define our main object of interest.

**Definition B.2.** The *spherical Fourier transform* of the Gelfand pair  $(G, K)$  is defined as the composition

$$\mathcal{F} : L^1(K \backslash G / K) \rightarrow C_0(\mathcal{S}_b(G, K)) \rightarrow C_0(\mathcal{S}^+(G, K)), \quad f \mapsto \widehat{f},$$

where the map  $L^1(K \backslash G / K) \rightarrow C_0(\mathcal{S}_b(G, K))$  is the Gelfand transform of the Banach algebra  $L^1(K \backslash G / K)$  and the second map is given by restriction to the subspace  $\mathcal{S}^+(G, K) \subset \mathcal{S}_b(G, K)$ .

Note that by definition,

$$\widehat{f}(\omega) = m_\omega(f) = \int_G f(x) \omega(x^{-1}) dm_G(x) \quad (f \in L^1(K \backslash G / K), \omega \in \mathcal{S}^+(G, K)). \quad (\text{B.2})$$

As in the abelian case, we have the usual Plancherel and inversion formulas.

**Lemma B.3** (Plancherel formula). *There exists a unique positive Radon measure  $\mu_{G, K}$  on  $\mathcal{S}^+(G, K)$  with the following properties.*

- (i) If  $f \in \mathcal{H}(G, K)$  and  $\widehat{f} \in L^1(\mathcal{S}^+(G, K), \mu_{G, K})$  then

$$f(g) = \int_{\mathcal{S}^+(G, K)} \widehat{f}(\omega) \omega(g) d\mu_{G, K}(\omega).$$

- (ii) If  $f \in L^1(K \backslash G / K) \cap L^2(K \backslash G / K)$ , then  $\widehat{f} \in L^2(\mathcal{S}^+(G, K), \mu_{G, K})$  and  $\|f\|_2 = \|\widehat{f}\|_2$ .
- (iii)  $\mathcal{F}$  extends to a bijective isometry  $L^2(K \backslash G / K) \rightarrow L^2(\mathcal{S}^+(G, K), \mu_{G, K})$ .

The measure  $\mu_{G, K}$  is called the *Plancherel measure* of the Gelfand pair  $(G, K)$ .

**B.6. The spherical Fourier transform for positive definite measures.** We can extend the spherical Fourier transform to positive definite Radon measures on  $K \backslash G / K$ . Here a Radon measure  $\eta$  on  $K \backslash G / K$  is called *positive definite* if  $\eta(f^* * f) \geq 0$  for every  $f \in C_c(G, K)$ . The following lemma is due to Godement [17].

**Lemma B.4** (Spherical Fourier transform of positive definite measures). *For every positive definite Radon measure  $\eta$  on  $K \backslash G / K$  there exists a unique positive Radon measure  $\widehat{\eta}$  on  $\mathcal{S}^+(G, K)$  such that the following hold.*

- (i) If  $f \in \mathcal{H}(G, K)$ , then  $\widehat{f} \in L^2(\mathcal{S}^+(G, K), \widehat{\eta})$ .
- (ii) The image  $\mathcal{F}(\mathcal{H}(G, K))$  is dense in  $L^2(\mathcal{S}^+(G, K), \widehat{\eta})$ .
- (iii) For all  $f_1, f_2 \in \mathcal{H}(G, K)$  we have

$$\eta(f_1^* * f_2) = \langle \widehat{f}_1, \widehat{f}_2 \rangle_{L^2(\mathcal{S}^+(G, K), \widehat{\eta})} = \int_{\mathcal{S}^+(G, K)} \overline{\widehat{f}_1(\omega)} \cdot \widehat{f}_2(\omega) d\widehat{\eta}(\omega).$$

The measure  $\widehat{\eta}$  is called the *spherical Fourier transform* of the measure  $\eta$ .

**B.7. Spherical decomposition of unitary representations.** Let  $(W, \rho)$  be a unitary representation of  $G$  which is completely reducible, i.e. decomposes as a direct sum of irreducibles. Given an irreducible unitary representation  $V$  of  $G$  we denote by  $W[V]$  the  $V$ -primary component of  $W$ , i.e. the sum of irreducible summands of  $W$  isomorphic to  $V$  so that

$$W = \bigoplus_{[V]} W[V],$$

where the sum is taken over all isomorphism classes of unitary  $G$ -representations occurring in  $W$ . We also define for every  $\omega \in \mathcal{S}^+(G, K)$  the corresponding eigenspace  $W_\omega \subset W^K$  by

$$W_\omega := \{w \in W \mid \forall \mu \in \mathcal{M}(G, K): \rho(\mu).w = \mu(\omega) \cdot w\}.$$

We then have the following well-known decomposition result.

**Lemma B.5** (Spherical decomposition, completely reducible case). *Let  $W$  be a completely reducible unitary  $G$ -representation. Then for every irreducible unitary  $G$ -representation  $V$  we have  $W[V]^K = W_{\omega_V}$ . In particular,*

$$W^K = \bigoplus_{\omega \in \mathcal{S}^+(G, K)} W_\omega.$$

*Proof.* It suffices to show that if  $V$  is an irreducible unitary  $G$ -representation, then  $\mathcal{M}(G, K)$  acts on  $V^K$  by  $\omega_V$ . Fix  $\mu \in \mathcal{M}(G, K)$ . Since  $V^K$  is one-dimensional and invariant under  $\mu$  there exists  $\lambda \in \mathbb{C}$  such that  $\rho(\mu).v = \lambda \cdot v$ . On the other hand we have

$$\mu(\omega_V) = \int_G \omega_V(g) d\mu(g) = \int_G \langle v, \rho(g).v \rangle d\mu(g) = \langle v, \rho(\mu).v \rangle = \lambda.$$

The lemma follows.  $\square$

In general, unitary  $G$ -representations need not be completely irreducible of course. However, if  $G$  admits a compact subgroup  $K$  such that  $(G, K)$  is a Gelfand pair, then  $G$  is automatically of type I (see e.g. [11, Thm. 2.2]) and hence its group  $C^*$ -algebra is postliminal ([12, Thm. 9.1]). In particular we can write every unitary  $G$ -representation  $(W, \rho)$  uniquely as a direct integral (over the unitary dual  $\widehat{G}$  of  $G$ ) of irreducible unitary representations (see e.g. [16, Thm. 7.32], [12, Thm. 8.6.6]). It follows that the  $\mathcal{H}(G, K)$ -representation  $(W^K, \rho)$  decomposes uniquely as a direct integral over the spherical part of the unitary dual, which we can identify with  $\mathcal{S}^+(G, K)$  by identifying each irreducible spherical representation with the associated spherical function. In view of [12, Prop. 8.6.8] we thus obtain:

**Lemma B.6** (Spherical decomposition, general case). *Let  $(W, \rho)$  be a unitary  $G$ -representation. Then there exists a measure  $\nu_W$  on  $\mathcal{S}^+(G, K)$  such that, as  $\mathcal{H}(G, K)$ -representations,*

$$W^K = \int_{\mathcal{S}^+(G, K)}^\oplus V_\omega d\nu_W(\omega),$$

where  $V_\omega$  is the spherical irreducible representation with associated spherical function  $\omega$ . The measure  $\nu_W$  is unique up to equivalence in the sense of [16, Thm. 7.32]. In particular, the atoms and the support of  $\nu_W$  depend only on  $W$  and the following hold.

- (i)  $\omega$  is an atom of  $\nu_W$  if and only if  $V_\omega$  is isomorphic to a subrepresentation of  $W$ .
- (ii)  $\omega \in \text{supp}(\nu_W)$  if and only if  $V_\omega$  is weakly contained in  $W$ .

**Definition B.7.** The support  $\text{spec}_{(G, K)}(W) := \text{supp}(\nu_W) \subset \mathcal{S}^+(G, K)$  is called the *spherical spectrum* of the representation  $(W, \rho)$ .

Note that  $(W, \rho)$  is completely reducible if and only if  $\nu_W$  is purely atomic and that in this case  $\text{supp}(\nu_W) = \{\omega \in \mathcal{S}^+(G, K) \mid W_\omega \neq \{0\}\}$  by Lemma B.5.

**B.8. The mean ergodic theorem for Gelfand pairs.** As in the previous sections  $G$  denotes a lcsc group and  $K < G$  a compact subgroup such that  $(G, K)$  is a Gelfand pair.

**Proposition B.8.** *Let  $(Y, \nu)$  be an ergodic  $G$ -space and let  $(\beta_t)$  be a sequence of bi- $K$ -invariant Borel probability measures on  $G$ . Then the following are equivalent.*

(i)  $(\beta_t)$  is weakly ergodic in the sense of Definition 5.3, i.e. for every  $f \in L^2(Y)^K$  the functions

$$F_t(y) := \int_G f(s^{-1}y) d\beta_t(s)$$

converge to the constant function  $\int_Y f d\nu$  with respect to the weak topology in  $L^2(Y)$  as  $t \rightarrow \infty$ .

(ii)  $\lim_{t \rightarrow \infty} \widehat{\beta_t}(\omega) = 0$  for  $\nu_{L^2(Y, \nu)}$ -almost all  $\omega \in \text{spec}_{(G, K)}(L^2(Y)) \setminus \{1\}$ .

*Proof.* We consider the subrepresentation  $W := L_o^2(Y) \subset L^2(Y)$ . Then

$$\text{spec}_{(G, K)}(L^2(Y)) \setminus \{1\} \subset \text{supp } \nu_W \subset \text{spec}_{(G, K)}(L^2(Y))$$

and since  $(Y, \nu)$  is ergodic, the trivial representation is not contained in  $W$ , and thus  $\nu_W(\{1\}) = 0$  by Lemma B.6. Observe that (i) is equivalent to the condition

$$\langle \pi_Y(\beta_t) f_1, f_2 \rangle_{L^2(Y)} \rightarrow 0 \quad (f_1, f_2 \in L_o^2(Y)^K),$$

which by polarization is equivalent to

$$\lim_{t \rightarrow \infty} \langle f, \pi_W(\beta_t) f \rangle = 0 \quad (f \in L_o^2(Y)^K). \quad (\text{B.3})$$

Under the isomorphism

$$W^K \cong \int_{\mathcal{S}^+(G, K)}^{\oplus} V_\omega d\nu_W(\omega)$$

the function  $f$  corresponds to a vector field  $(f_\omega)$  on  $\text{supp}(\nu_W)$  and we have

$$\pi_W(\beta_t) \cdot (f_\omega) = \widehat{\beta_t}(\omega) \cdot f_\omega.$$

Thus (B.3) is equivalent to  $\lim_{t \rightarrow \infty} \widehat{\beta_t}(\omega) = 0$  for  $\nu_W$ -almost all  $\omega$ , or equivalently,  $\nu_{L^2(Y, \nu)}$ -almost all  $\omega \in \text{spec}_{(G, K)}(L^2(Y)) \setminus \{1\}$ . This shows that (i) and (ii) are equivalent.  $\square$

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